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Modified Differential Transform Method for Solving Vibration Equations of MDOF Systems

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Abstract

Vibration equations of discrete multi-degrees-of-freedom (**MDOF**) structural systems is system of differential equations. In linear systems, the differential equations are also linear. Various analytical and numerical methods are available for solving the vibration equations in structural dynamics. In this paper modified differential transform method (**MDTM**) as a semi-analytical approach is generalized for the system of differential equations and is utilized for solving the vibration equations of **MDOF** systems. The **MDTM** is a recursive method which is a hybrid of Differential Transform Method (**DTM**), Pade' approximant and Laplace Transformation. A series of examples including forced and free vibration of **MDOF** systems with classical and non-classical damping are also solved by this method. Comparison of the results obtained by **MDTM** with exact solutions shows good accuracy of the proposed method; so that in some cases the solutions of the vibration equation that found by **MDTM** are the exact solutions. Also, **MDTM** is less expensive in computational cost and simpler with compare to the other available approaches.

Keywords: Modified Differential Transform Method; Multi-Degrees-of-Freedom Systems; Pade' Approximant; Vibration Equation.

1. Introduction

Most of the existing structural systems are multi-degrees-of-freedom (**MDOF**) that can be categorized into discrete and continuous systems. In discrete systems, the structural properties such as mass, stiffness and damping are localized and the equations of motion are in the form of a system of ordinary differential equations. On the other hand in continuous systems the structural properties are distributed and the vibration equations are in the form of partial differential equations. Discrete systems have finite number of degrees of freedom while in continuous systems the number of degrees of freedom is infinite. There are different tools such as the generalized coordinates method which can be applied on continuous systems to formulate the equations of motion. By these methods the vibration equation of a continuous system turns into the vibration equation of a discrete system with finite degrees of freedom. Hence, solving the vibration equations of discrete **MDOF** systems has been considered significantly by researchers.

Vibration equation of a linear discrete structural system with initial values is as follows:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = p(t) \qquad u(0) = a \qquad \dot{u}(0) = b \tag{1}$$

In the case of linear **MDOF** systems, Eq. (1) is a system of linear ordinary differential equations and *u(t)* is the displacement vector along degrees of freedom. Also, *m* is the mass matrix, *c* is the damping matrix, *k* is the stiffness matrix and *p(t)* is the applied load vector on the system. The left hand side of Eq. (1) consists of inertia force ($m\ddot{u}(t)$), damping force ($c\dot{u}(t)$) and elastic force *(k(u(t))* and the right hand side of the equation is the applied load vector. In Figure (1-a) a discrete **MDOF** system is illustrated. In single-degree-of-freedom (**SDOF**) systems, Eq. (1) turns into a

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linear ordinary differential equation and the coefficients of the equation are single numbers instead of matrices. The mass-spring model in Figure (1-b) is an example of a **SDOF** system.

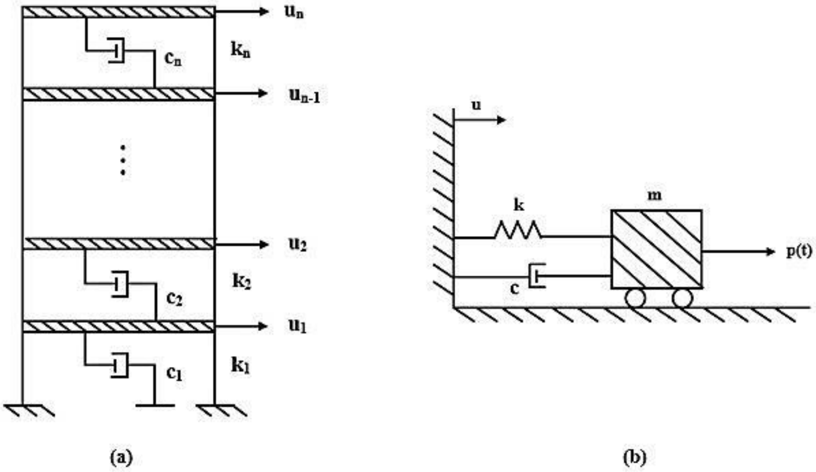

Figure 1. Discrete structural systems, a- a MDOF system, b- a SDOF system

Solving the vibration equation of structural systems (both **SDOF** and **MDOF** systems) has been addressed from past decades by many researchers. Hence, different analytical and numerical approaches have been introduced for this purpose. Direct Integration Method has been used widely for solving the vibration equations. For this purpose, various numerical methods such as Newmark β Method and Wilson θ method have been employed for numerical integration of the equations of motion. Extensive information about these methods is available in the structural dynamics text books, [1, 2].

Recently, Differential Transform Method (**DTM**) as a recursive semi-analytical approach for solving initial value problems has been introduced. This method was first proposed by Zhou [3] (1986) for solving linear and non-linear initial value problems in electric circuit analysis. Thereafter this method has been developed by other researchers [4-9]. Since vibration equation of a **SDOF** system is a differential equation with initial values, **DTM** has been employed for solving vibration equation of linear and nonlinear **SDOF** oscillators in structural dynamics. El. Shahed [10] applied **DTM** for solving vibration equation of non-linear **SDOF** oscillatory systems. He proposed the modified differential transform method (**MDTM**) with Pade Approximation for this purpose. Momania and Erturk [11] also applied **MDTM** for solving vibration equation of non-linear **SDOF** oscillators. They concluded that **MDTM** is an efficient method for calculating periodic solutions of non-linear oscillatory systems. The examples indicated that **MDTM** greatly improves **DTM**'s truncated series solution in the convergence rate, and it often yields the true analytic solution. Nourazar and Mirzabeigy [12] employed **MDTM** for solving the nonlinear Duffing oscillator with damping effect. They also found the solution by using the fourth-order Runge–Kutta numerical method. Comparison of the results showed good accuracy of **MDTM** results. **DTM** has been also employed for solving the free vibration equations of continuous systems such as beams and plates. As one of the earliest works, Chen and Ho [13] applied **DTM** for solving transverse vibration of a rotating twisted Timoshenko beam under axial loading. They found closed form solutions for the free vibration problems of a rotating twisted Timoshenko beam. In another research, they applied **DTM** for solving the free and forced vibration problems of general elastically end restrained non-uniform beams resting on a non-homogeneous elastic foundation and subjected to axial tensile and transverse forces [14]. Ozdemir and Kaya [15] utilized **DTM** for bending vibration analysis of a rotating tapered cantilever Bernoulli–Euler beam. They found the non-dimensional angular velocities of the beams. Also the effects of different parameters on vibration characteristics of the rotating tapered cantilever beam were evaluated. Catal [16] used **DTM** for free vibration equations of a beam on elastic soil. It was concluded that the rate of convergence and accuracy of **DTM** was very good. Vibration of composite sandwich beams with visco-elastic core was also investigated by application of **DTM** by Arikoglu and Ozkol [17]. Demirdag and Yesilce [18] applied **DTM** for free vibration analysis of an elastically supported Timoshenko column with a tip mass. Comparison of the results from **DTM** with the exact solution showed very good accuracy of the **DTM** results. The method is also employed for vibration of plates. Recently, Lal and Ahlawat [19] applied **DTM** for axisymmetric vibration and buckling analysis of functionally graded circular plates subjected to uniform in-plane forces.

As it is seen, numerous researches on application of **DTM** and **MDTM** for solving the vibration equation of **SDOF** systems has been carried out, where the equation of motion is a single differential equation. Based on the ability of **MDTM** for solving the vibration equations of oscillatory systems and according to the interests in solving the vibration equations of **MDOF** systems, in current research **MDTM** is applied on the vibration equations of **MDOF** systems. In this paper, at first **DTM** is introduced for solving the differential equations with initial values. Since this method gives the solutions in restricted intervals, it is not suitable for solving the vibration equations of oscillatory systems. Therefore **MDTM** which is a hybrid of **DTM**, Pade' approximant and Laplace transformation is described and the procedure for solving vibration equations of **MDOF** systems is presented. Finally, some examples which include free and forced vibration of damped and un-damped systems subjected to different loading functions are solved by **MDTM** and the results are compared with the other common methods.

2. Differential Transform Method

Function *u(t)* which is analytical in domain Ω can be written in the form of a power series around each arbitrary point in this domain. Differential transform of function *u(t)* is given by:

$$U(k)=\frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=0}. \tag{2}$$

In Eq. (2), *u(t)* and *U(k)* are original and transformed functions, respectively. Differential inverse transform of *U(k)* is defined as follows:

$$u(t)=\sum_{k=0}^{\infty}U(k)t^k \tag{3}$$

Combining Eqs. (2) and (3), yields the following equation for *u(t)*:

$$u(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=0}. \tag{4}$$

In typical applications, function *u(t)* is represented by a finite number of terms. Obviously, the accuracy will be improved by increasing number of terms. Therefore, Eq. (4) can be rewritten as follows:

$$u(t)=\sum_{k=0}^{N}U(k)t^k. \tag{5}$$

Some transformation rules for differential transform method which are useful for practical problems are stated as follows:

1) If $z(t)=u(t)\pm v(t)$ then $Z(k)=U(k)\pm V(k)$.
2) If $z(t)=\alpha u(t)$ then $Z(k)=\alpha U(k)$ where α is a constant.
3) If $z(t)=\dfrac{du(t)}{dt}$ then $Z(k)=(k+1)U(k+1)$.
4) If $z(t)=\dfrac{d^2u(t)}{dt^2}$ then $Z(k)=(k+1)(k+2)U(k+2)$.
5) If $z(t)=\dfrac{d^mu(t)}{dt^m}$ then $Z(k)=(k+1)\cdots(k+m)U(k+m)$.
6) If $z(t)=u(t)v(t)$ then $Z(k)=\sum_{l=0}^{k}U(l)V(k-l)$.
7) If $z(t)=t^m$ then $Z(k)=\delta(k-m)=\begin{cases}1, & if\ \ k=m\\ 0, & if\ \ k\neq m\end{cases}$.
8) If $z(t)=\exp(\lambda t)$ then $Z(k)=\dfrac{\lambda^k}{k!}$ where λ is a constant.
9) If $z(t)=(1+t)^m$ then $Z(k)=\dfrac{m(m-1)\dots(m-k+1)}{k!}$.
10) If $z(t)=\cos(\omega t+\alpha)$ then $Z(k)=\dfrac{\omega^k}{k!}\cos(\dfrac{\pi k}{2}+\alpha)$ where ω and α are constants.
11) If $z(t)=\sin(\omega t+\alpha)$ then $Z(k)=\dfrac{\omega^k}{k!}\sin(\dfrac{\pi k}{2}+\alpha)$ where ω and α are constants.

3. Differential Transform Method for Vibration Equation of MDOF Systems

DTM can be applied for solving systems of differential equations with initial values. As stated before, vibration equation of a discrete **MDOF** system is in the form of a system of differential equations with initial values. Therefore, **DTM** can be utilized for solving vibration equation of these systems. The vector form of the vibration equation of a discrete **MODF** system is as follows:

$$\mathrm{M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = p(t),\ u(0) = A,\ \dot{u}(0) = B}\ . \tag{6}$$

Where the mass matrix (*M*), the damping matrix (*C*) and the stiffness matrix (*K*) are $n\times n$ matrices and displacements along degrees of freedom (*u(t)*) and applied loads (*p(t)*) are $n\times 1$ vectors. *n* represents the number of degrees of freedom of the system. *A* and *B* are also the vector of initial displacements and velocities, respectively. The general form of the vectors *u(t)* and *p(t)* are as follows:

$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix},\ \ p(t) = \begin{bmatrix} p_1(t) \\ \vdots \\ p_n(t) \end{bmatrix}. \tag{7}$$

Eq. (6) is a system of *n*-ordinary differential equations with initial values where unknowns are $u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}$.

To solve Eq. (6) by **DTM**, first of all differential transform is applied on the system of differential equations and their initial values. Consequently, a new system of equations will be obtained as follows:

$$M(k+1)(k+2)U(k+2) + C(k+1)U(k+1) + KU(k) = \frac{1}{k!}\left[\frac{d^k p(t)}{dt^k}\right]_{t=.0}. \tag{8}$$

$$U(0) = A\ ,\ \ U(1) = B\ . \tag{9}$$

Where $U(k+2) = \begin{bmatrix} U_1(k+2) \\ \vdots \\ U_n(k+2) \end{bmatrix}$, $U(k+1) = \begin{bmatrix} U_1(k+1) \\ \vdots \\ U_n(k+1) \end{bmatrix}$, $U(k) = \begin{bmatrix} U_1(k) \\ \vdots \\ U_n(k) \end{bmatrix}$, $U_i(k)$ for $i=1,2,3,\ldots,n$ are differential transforms of functions $u_i(t)$ and $\frac{d^k p(t)}{dt^k} = \begin{bmatrix} \frac{d^k p_1(t)}{dt^k} \\ \vdots \\ \frac{d^k p_n(t)}{dt^k} \end{bmatrix}$.

Consider *U(k)*, *U(k+1)* and *U(k+2)* as new unknowns of the system of Eq. (8). According to initial values presented in Eq. (9), a recursive relation is obtained from system of Eq. (8) for different values of *k*. Note that $U_i(k)$ $i=1,2,\ldots,n$ for different values of *k* are coefficients of power expansion of function $u_i(t)$. By application of **DTM**, system of differential equations (6) turns into a series of simple algebraic systems of equations.

In the system of Eq. (8), let $P(k) = \frac{1}{k!}\left[\frac{d^k p(t)}{dt^k}\right]_{t=0}$, therefore recursive relation can be written as follows:

$$M(k+1)(k+2)U(k+2) + C(k+1)U(k+1) + KU(k) = P(k). \tag{10}$$

Since *p(t)* is a known function, *P(k)* is also a known vector for different values of *k*. In each step of process, for given values of *k*, *U(k)* and *U(k+1)*are known values which were obtained from the previous steps, except for *k=0* that in this case *U(k)* and *U(k+1)* are konwn from initial values (9). Therefore, in each step of the recursive process, only *U(k+2)* is unknown in the system of Eq. (10).

Recursive relations that are obtained from Eq. (10) for different values of *k* are as follows:

$$k=0:\quad 2MU(2) + CU(1) + KU(0) = P(0). \tag{11}$$

$$k=1:\quad 6MU(3) + 2CU(2) + KU(1) = P(1). \tag{12}$$

$$k=2:\quad 12MU(4) + 3CU(3) + KU(2) = P(2). \tag{13}$$

……

Recursive relations in Eqs. (11-13) can be solved successively and consequently the coefficients of the power series of *u(t)* would be determined. Finally, by applying Differential Inverse Transform, *u(t)* would be obtained.

4. Pade' Approximants

Based on differential transform method (**DTM**) which was described before, in order to achieve an accurate analytical response of vibration equation, Pade' Approximant should be applied on the truncated Taylor expansion which is obtained from **DTM**. Modification of **DTM** by Pade' Approximant was suggested first by El Shahed [10] for a **SDOF** system.

Pade' Approximant is a ratio of two polynomials which is made by Taylor expansion of a function *y(x)* [20]. The $\left[\frac{L}{M}\right]$ Pade' Approximant of function *y(x)* is given by:

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)} \quad . \tag{14}$$

Where $P_L(x)$ and $Q_M(x)$ are polynomials of degrees at most L and M, respectively. Coefficients of the polynomials $P_L(x)$ and $Q_M(x)$ are determined by the following power series:

$$y(x) = \sum_{i=1}^{\infty} a_i x^i \tag{15}$$

$$y(x) - \frac{P_L(x)}{Q_M(x)} = o(x^{L+M+1}) \tag{16}$$

Since the numerator and denominator can be multiplied by a constant and $\left[\frac{L}{M}\right]$ remains unchanged, the following normalization condition is applied:

$$Q_M(0) = 1.0 \quad . \tag{17}$$

It is required that $P_L(x)$ and $Q_M(x)$ don't have any common factors. $P_L(x)$ and $Q_M(x)$ can be represented as follow:

$$\begin{aligned} P_L(x) &= p_0 + p_1x^1 + p_2x^2 + \cdots + p_Lx^L, \\ Q_M(x) &= q_0 + q_1x^1 + q_2x^2 + \cdots + q_Mx^M, \end{aligned} \tag{18}$$

Then, based on Eqs. (17) and (18), Eq. (16) can be multiplied by $Q_M(x)$ to linearize the coefficients of equations. Finally, Eq. (16) can be obtained as follows:

$$\begin{aligned} a_{L+1} + a_Lq_1 + \cdots + a_{L-M+1}q_M &= 0, \\ a_{L+2} + a_{L+1}q_1 + \cdots + a_{L-M+2}q_M &= 0, \\ &\vdots \\ a_{L+M} + a_{L+M-1}q_1 + \cdots + a_Lq_M &= 0, \end{aligned} \tag{19}$$

$$\left.\begin{aligned} a_0 &= p_0 \\ a_1 + a_0q_1 &= p_1 \\ a_2 + a_1q_1 + a_0q_2 &= p_2 \\ &\vdots \\ a_L + a_{L-1}q_1 + \cdots a_0q_L &= p_L \end{aligned}\right\} \tag{20}$$

Therefore, at first, q's unknowns are obtained from Eq. (19). Then, p's unknowns are obtained from Eq. (20) by knowing q's.

If Eqs. (19) and (20) are nonsingular, then they can be solved directly as shown in Eq. (21) [20]. In Eq. (21), if lower index on a sum exceeds upper index, sum is replaced by zero.

$$\left[\frac{L}{M}\right] = \frac{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M}x^j & \sum_{j=M-1}^{L} a_{j-M+1}x^j & \cdots & \sum_{j=0}^{L} a_j x^j \end{bmatrix}}{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{bmatrix}}. \quad (21)$$

5. Modified Differential Transform Method

Truncated power series which obtained from **DTM** is an approximate solution of the vibration equation. Usually, the convergence domain of this power series for vibration problems is very small. So that, it is only accurate in a very small time interval and it is not appropriate for larger time intervals. Using a hybrid of **DTM**, Pade' approximant and Laplace transformation leads to increasing the convergence domain of the solution [10]. In this way **DTM** is modified and is called Modified Differential Transform Method (**MDTM**) in literature. In most cases the solution which obtained by **MDTM** might be the exact solution. The procedure of the **MDTM** is explained in subsequent paragraphs.

For each entry of vector *u(t)* which is a truncated power series and was obtained previously by **DTM**, the following steps should be applied:

1) First, Laplace transform is applied on each entry of vector *u(t)*.
2) s is substituted by $\frac{1}{t}$ in resulting expressions from step 1.
3) Pade' approximant of order $\left[\frac{L}{M}\right]$ is applied on transformed series which were obtained in step 2. L and M are arbitrary and should be smaller than the order of the power series. In this step, Pade' approximant leads to improvement of the solution accuracy and increases convergence domain of the solution.
4) t is substituted by $\frac{1}{s}$.
5) Finally, inverse Laplace transform is applied on resulting expression in respect of variable s. Consequently, this method leads to an approximate or accurate solution of the vibration Equation.

6. Examples

To evaluate the efficiency of **MDTM** for solving vibration equations of **MDOF** systems, a series of examples are solved by this method. The examples include free and forced vibration of structural systems with or without damping. The solutions obtained from **MDTM** are compared with the solutions that result from **DTM** and Fehlberg fourth-fifth order Runge-Kutta (**RKF45**) as a numerical method for solving systems of differential equations.

6.1. Forced Vibration of a Damped SDOF System

In this example a damped **SDOF** system subjected to an external load is considered. Vibration equation of the system is given in Eq. (22):

$$0.1\ddot{u} + 350u = 7.2(e^{-10t} - e^{-100t}), \qquad u(0) = 0, \qquad \dot{u}(0) = 0. \quad (22)$$

Here, **DTM** and **MDTM** are applied for solving the vibration equation and the solutions are compared with the exact one.

The procedure of solving the above vibration equation by **DTM** is as follows:

At first, differential transform is applied on both sides of Eq. (22) and its initial values:

$$0.1 \times U(k+2) + 350 \times U(k) = 7.2 \times (\frac{(-10)^k}{k\ !} - \frac{(-100)^k}{k\ !})\ ,$$
$$U(0) = 0, \quad U(1) = 0\ ,$$
$$for \quad k = 0,1,2,3,\ldots \quad . \tag{23}$$

Eq. (23) turns into the following recursive relation by a slight simplification:

$$U(k+2) = -3500 \times U(k) - 72 \times (\frac{(-10)^k}{k\ !} - \frac{(-100)^k}{k\ !})\ ,$$
$$U(0) = 0, \quad U(1) = 0\ ,$$
$$for \quad k = 0,1,2,3,\ldots \quad . \tag{24}$$

As described in section 3, *U(k)* which contains coefficients of power series of *u(t)*, can be calculated by using recursive relation (24). Thus, the truncated power series of *u(t)* is abtained as follows:

$$u(t) = 1080.0\ t^3 - 29700.0\ t^4 + 4.104 \times 10^5 t^5 - 6.534 \times 10^6 t^6 + 1.086557143 \times 10^8 t^7\ . \tag{25}$$

Here seven terms of the series are considered. Eq. (25) is plotted in Figure (2).

Now, the results from **DTM** can be improved by **MDTM**. Therefore, Laplace transform is applied on both sides of Eq. (25) as follows:

$$L[u(t)] = 72 \times (90 \frac{1}{s^4} - 9900 \frac{1}{s^5} + 6.84000 \times 10^5 \frac{1}{s^6} - 6.5340000 \times 10^7 \frac{1}{s^7} + 7.605900001 \times 10^9 \frac{1}{s^8} \quad . \tag{26}$$

By applying **MDTM,** *s* is substituted by $\frac{1}{t}$ in Eq. (26) and Pade' approximant of order $\left[\frac{4}{4}\right]$ is calculated. Then, *t* is substituted by $\frac{1}{s}$ and the following equation is obtained:

$$\left[\frac{4}{4}\right] = \frac{6479.9999}{(0.9999\ \mathrm{s}^4 + 110.0000000\ \mathrm{s}^3 + 4499.999998\ \mathrm{s}^2 + 3.850000000 \times 10^5\ \mathrm{s} + 3.500000041 \times 10^6)}, \tag{27}$$

Now, inverse Laplace transform is applied on Eq. (27) and solution of the vibration equation is determined as follows:

$$u(t) = -0.0053\ \exp(-100t) + 0.02\ \exp(-10.0\ t) -$$
$$\exp(5.2935 \times 10^{-8} t) \times (0.0147 \times \cos(59.1608t) + 0.0056 \times \sin(59.1608t)) \quad . \tag{28}$$

Exact solution of Eq. (22) is also available [1] and is as follows:

$$u(t) = -0.0056 \sin(59.16 \times t) - 0.01467 \cos(59.16 \times t) + 0.02 \exp(-10t) - 0.0053 \exp(-100t)\ . \tag{29}$$

It is seen that **MDTM** can produce an analytical solution for Eq.(22).

In Figure (2-a), the diagrams of function *u(t)* which result from **MDTM** and the exact solution are plotted. Comparison of the curves shows a remarkable accuracy of **MDTM**. In Figure (2-b), the diagrams of function *u(t)* which result from **DTM**, **MDTM** and the exact solution are plotted. Comparison of the curves shows that **DTM** is only accurate in a very small time interval and this method is not applicable for these problems.

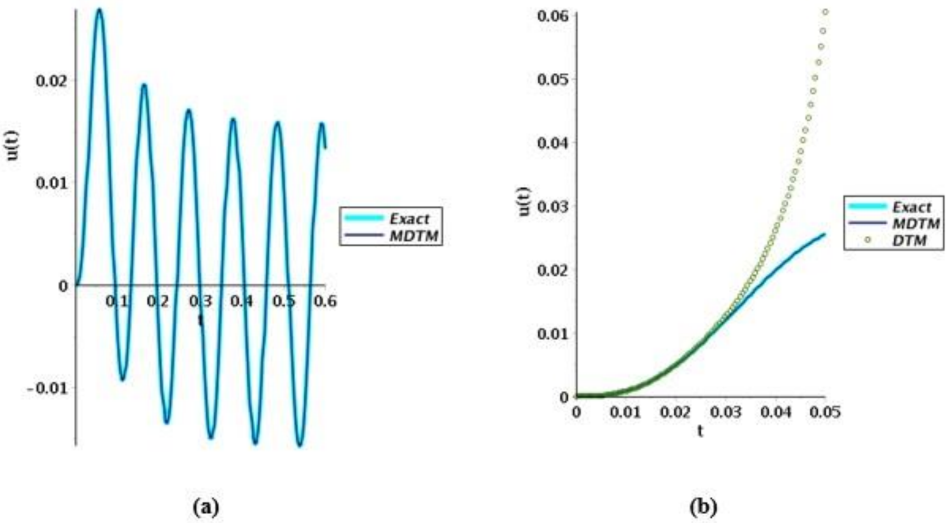

Figure 2. Response of the SDOF system, a- exact and MDTM b- exact, MDTM and DTM

6.2. Free Vibration of an Un-damped Two DOF System

In this example, free vibration of a two **DOF** system without damping is considered. Vibration equation of the system and the mass and stiffness matrices are given below:

$$M\ \ddot{u} + Ku = 0, \qquad \mathrm{u}(0) = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \qquad \dot{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad . \tag{30}$$

Where $M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $K = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$.

Here, **DTM** and **MDTM** are applied for solving the vibration equation. Also, solution of the equation is obtained from **RKF45** which is a useful numerical method for solving systems of differential equations. Finally, the results are compared with each other.

The procedure for solving the above vibration equation by **DTM** is as follows:

Eq. (30) is equivalent to the following system of differential equations with initial values as is stated here:

$$\begin{aligned} &2\ \ddot{u}_1 + 3\ u_1 - u_2 = 0 \\ &\ddot{u}_2 - u_1 + u_2 = 0 \\ &u_1(0) = \frac{1}{2}, \quad u_2(0) = 1, \quad \dot{u}_1(0) = 0, \quad \dot{u}_2(0) = 0, \end{aligned} \tag{31}$$

At first, differential transform is applied on all the differential equations of Eqs. (31) and the initial values:

$$\begin{aligned} &2\ (\mathrm{k}+2)\,U_1(k+2) + 3\ U_1(k) - U_2(k) = 0\ , \\ &U_2(k+2) - U_1(k) + U_2(k) = 0\ , \\ &U_1(0) = \frac{1}{2}, \quad U_2(0) = 1, \quad U_1(1) = 0, \quad U_2(1) = 0\ , \\ &\text{for } \mathrm{k} = 0,1,2,3,.... \quad . \end{aligned} \tag{32}$$

The system of Eqs. (32) turns into the following recursive relations by a slight simplification:

$$\begin{aligned} &U_1(k+2) = \frac{-3\ U_1(k) + U_2(k)}{2\ (\mathrm{k}+2)}\ , \\ &U_2(k+2) = U_1(k) - U_2(k)\ , \\ &U_1(0) = \frac{1}{2}, \quad U_2(0) = 1, \quad U_1(1) = 0, \quad U_2(1) = 0\ , \\ &\text{for } \mathrm{k} = 0,1,2,3,.... \quad . \end{aligned} \tag{33}$$

As it was mentioned in section 3, by using recursive relations in Eqs. (33) and solving the system of equations in each step, $U_1(k)$ and $U_2(k)$ which contain coefficients of power series of the functions $u_1(t)$ and $u_2(t)$, can be obtained. Therefore, the truncated power series of functions $u_1(t)$ and $u_2(t)$ are determined as follows:

$$u_1 = 0.5 - 0.0125t^2 + 0.0052t^4 - 0.0001t^6 ,$$
$$u_2 = 1 - 0.25t^2 + 0.0104t^4 - 0.0002t^6 \quad . \tag{34}$$

Here seven terms of the series are considered. It should be noted that even if the number of terms of the truncated power series of $u_1(t)$ and $u_2(t)$ are increased, the convergence domain of functions $u_1(t)$ and $u_2(t)$ which result from **DTM** wouldn't be improved. Therefore, **MDTM** is employed. For each of the functions $u_1(t)$ and $u_2(t)$, the following steps would be applied. These steps will be described with more details for $u_1(t)$.

Laplace transform is applied on $u_1(t)$ as follows:

$$L[u_1(t)] = 0.5\frac{1}{s} - 0.25\frac{1}{s^3} + 0.1248\frac{1}{s^5} - 0.072\frac{1}{s^7} \quad . \tag{35}$$

By applying **MDTM**, s is substituted by $\frac{1}{t}$ in Eq. (35) and then Pade' approximant of order $\left[\frac{3}{3}\right]$ is calculated. Then t is substituted by $\frac{1}{s}$ and the following equation is obtained:

$$\left[\frac{3}{3}\right] = \frac{0.5s^2 - 0.0004}{s^3 + 0.4992s} , \tag{36}$$

Finally, inverse Laplace transform is applied on the Eq. (36) and the solution of the vibration equation is obtained as follows:

$$u_1(t) = 0.5008 \times \cos(0.7065t) - 0.0016 \quad . \tag{37}$$

It is seen that **MDTM** has produced an analytical solution for Eq. (30).

In Figure (3-a), the diagrams of function $u_1(t)$ which result from **MDTM** and **RKF45** method are plotted. Comparison of the curves shows a remarkable accuracy of **MDTM**. In Figure (3-b), the diagrams of function $u_1(t)$ which result from **DTM**,**MDTM** and **RKF45** are plotted. Comparison of the plotting curves shows that **DTM** is only accurate in a very small time interval and again it is not applicable for vibration problems.

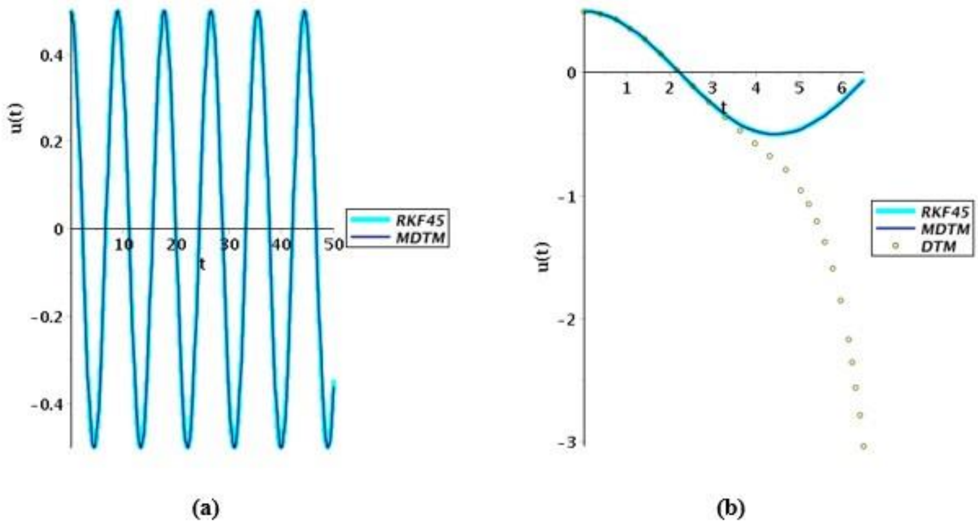

Figure 3. Response of the un-damped two DOF system (u1(t)), a- RKF45 and MDTM, b- RKF45, MDTM and DTM

The above approach is also applied on function $u_2(t)$ and the final result is stated here. Function $u_2(t)$ which results from **MDTM** and Pade' approximant of order $\left[\frac{3}{3}\right]$ is as follows:

$$u_2(t) = 1.0016 \times \cos(0.7065t) - 0.0016 \quad , \tag{38}$$

Results which obtained for function $u_2(t)$ are plotted in Figures (4-a) and (4-b). Comparison of the curves shows that the proposed method in this article has an acceptable accuracy.

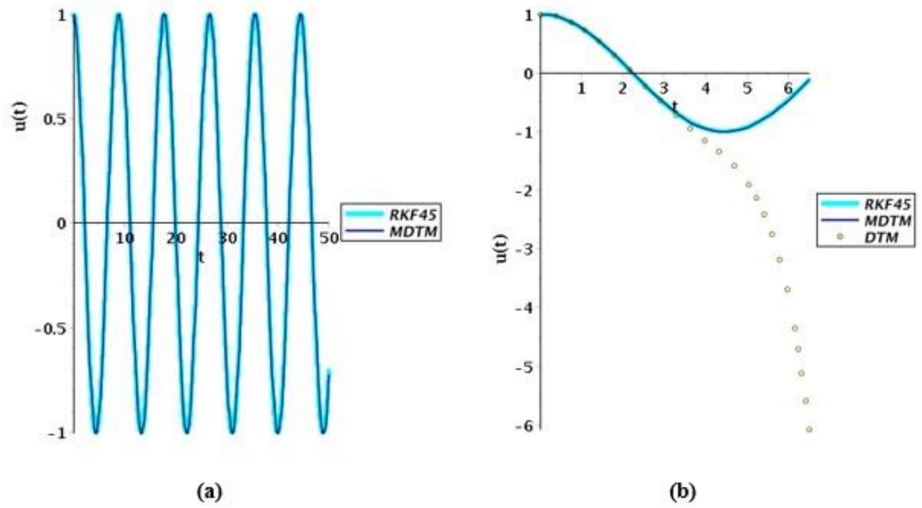

Figure 4. Response of the un-damped two DOF system (u2(t)), a- RKF45 and MDTM b- RKF45, MDTM and DTM

As it is seen again, **MDTM** produces an acceptable solution for the vibration equations. While in the case of **DTM**, the convergence domain is not in any way appropriate.

6.3. Forced Vibration of an Un-damped Two DOF System

In this example an un-damped two **DOF** system subjected to a harmonic dynamic load is considered. The forced vibration equation of the system is given below:

$$M\ \ddot{u} + Ku = p(t), \qquad \mathrm{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \dot{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{39}$$

Where $M = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}$, $K = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$ and $p(t) = \begin{bmatrix} 100 \times \sin(0.8\ t) \\ 0 \end{bmatrix}$.

Similar to the previous example, at first **MDTM** is applied on the Eq. (39). Then the resulting solution is compared with the solution obtained from **RKF45**.

Eq. (39) is equivalent to the following system of differential equations with initial values:

$$\begin{aligned} &0.2\ddot{u}_1 + 3u_1 - u_2 = 100 \times \sin(0.8\ t)\ , \\ &0.1\ddot{u}_2 - u_1 + u_2 = 0\ , \\ &u_1(0) = 0, \quad u_2(0) = 0, \quad \dot{u}_1(0) = 0, \quad \dot{u}_2(0) = 0\ . \end{aligned} \tag{40}$$

Now **MDTM** and Pade' approximant of order $\left[\frac{7}{6}\right]$ which were described completely in previous examples are applied on the Eq. (40) and its solution is obtained as follows:

$$\begin{aligned} u_1 &= 55.4058 \times \sin(0.8019t) - 13.7203 \times \sin(2.2344t) - 3.0806 \times \sin(4.4720t)\ , \\ u_2 &= 59.2777 \times \sin(0.7979t) - 27.3013 \times \sin(2.2371t) + 3.0809 \times \sin(4.4720t)\ . \end{aligned} \tag{41}$$

It is seen that **MDTM** produces an analytic solution for the vibration equation.

In Figure (5-a), the diagrams of function $u_1(t)$ which result from **MDTM** and **RKF45** method are plotted. Comparison of the plotting curves shows a remarkable accuracy of **MDTM**. In Figure (5-b), the diagrams of the function $u_1(t)$ which result from **DTM**, **MDTM** and **RKF45** are plotted. Comparison of the curves shows that **DTM** is only accurate in a very small time interval.

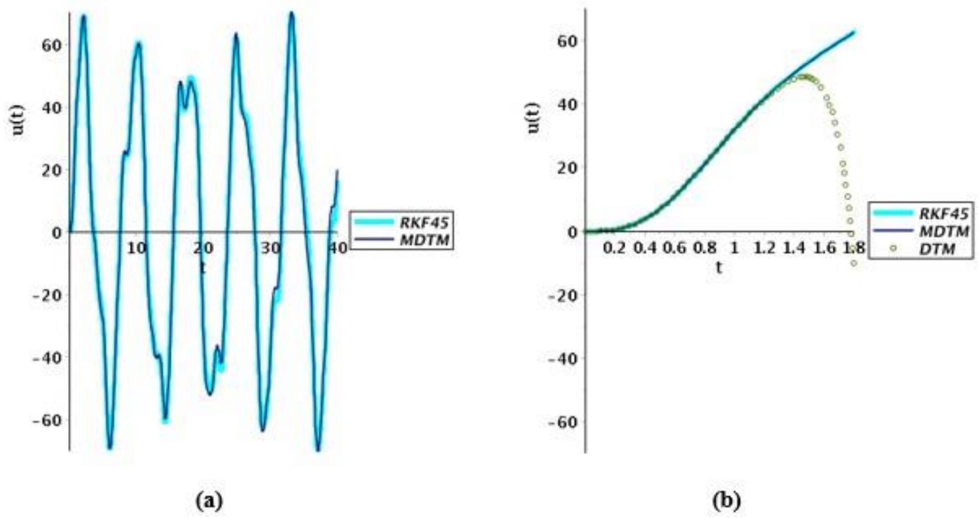

Figure 5. Response of the un-damped two DOF system (u1(t)), a- RKF45 and MDTM b- RKF45, MDTM and DTM

Results which obtained for function $u_2(t)$ are plotted in Figure (6). Comparison of the figures shows that the proposed method in this paper has an acceptable accuracy.

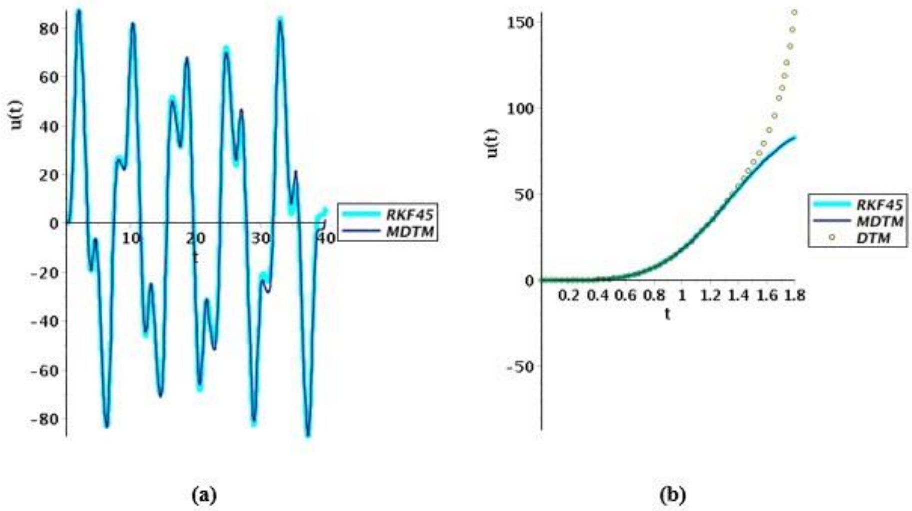

Figure 6. Response of the un-damped two DOF system (u2(t)), a- RKF45 and MDTM b- RKF45, MDTM and DTM

6.4. Forced Vibration of Two DOF System with Classical Damping

To show that the proposed method is applicable to the problems with damping, a two **DOF** system with classical damping is taken into consideration here. The forced vibration equation of the system is given below:

$$M\ \ddot{u} + C\ \dot{u} + Ku = p(t), \qquad \mathrm{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \dot{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{42}$$

Where $M = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}$, $C = \begin{bmatrix} 0.1344 & -0.0448 \\ -0.0448 & 0.0448 \end{bmatrix}$, $K = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$ and $p(t) = \begin{bmatrix} 100 \times \sin(0.8\ t) \\ 0 \end{bmatrix}$.

Eq. (42) is equivalent to the following system of differential equations with initial values:

$$\begin{aligned} &0.2\ddot{u}_1 + 0.1344\dot{u}_1 - 0.0448\dot{u}_2 + 3\ u_1 - u_2 = 100 \times \sin(0.8\ t)\ , \\ &0.1\ddot{u}_2 - 0.0448\dot{u}_1 + 0.0448\dot{u}_2 - \ u_1 + u_2 = 0\ , \\ &u_1(0) = 0, \quad u_2(0) = 0, \quad \dot{u}_1(0) = 0, \quad \dot{u}_2(0) = 0\ \ . \end{aligned} \tag{43}$$

MDTM and Pade' approximant of order $\left[\frac{6}{6}\right]$ is applied on the vibration equation and the corresponding solution is as follows:

$$\begin{aligned} u_1 &= \exp(-0.4480t)\times(0.6366\times\cos(4.4496t)-3.0277\times\sin(4.4496t)) \\ &\quad+\ \exp(-0.1117t)\times(1.5702\times\cos(2.2318t)-13.6310\times\sin(2.2318t)) \\ &\quad+\exp(-0.0001t)\times(-2.2068\times\cos(0.8018t)+55.3216\times\sin(0.8018t))\ , \\ u_2 &= \exp(-0.4480t)\times(-0.6363\times\cos(4.4496t)+3.0273\times\sin(4.4496t)) \\ &\quad+\ \exp(-0.1122t)\times(3.1447\times\cos(2.2333t)-27.1798\times\sin(2.2333t)) \\ &\quad+\exp(-0.0002t)\times(-2.5084\times\cos(0.7999t)+59.1304\times\sin(0.7999t))\ . \end{aligned} \tag{44}$$

It is seen that **MDTM** produces an analytic solution for Eq. (42).

In Figure (7), the diagrams of functions $u_1(t)$ and $u_2(t)$ which result from **MDTM** and **RKF45** method are plotted. Comparison of the plotting curves shows remarkable accuracy of **MDTM** for the response of the both **DOF.**

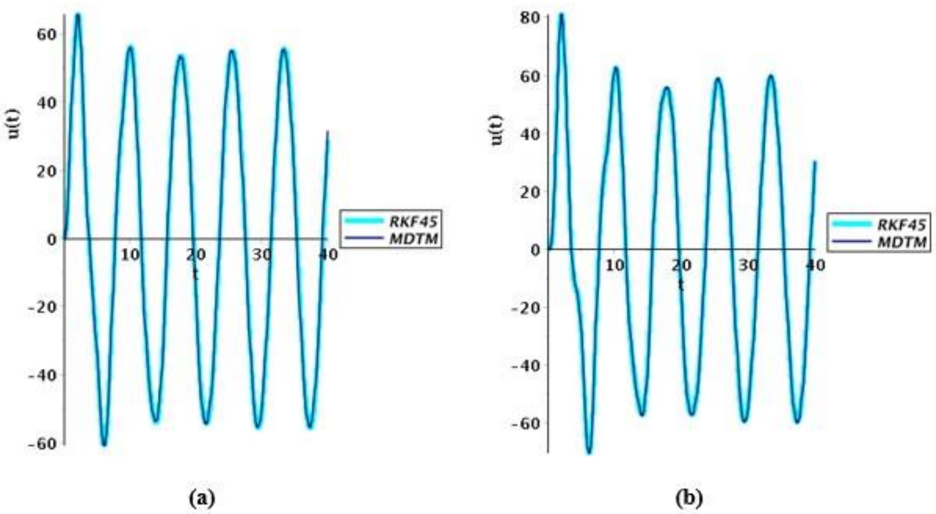

Figure 7. Response of the damped two DOF system a- u1(t), b- u2(t)

6.5. Forced Vibration of a Two DOF System with Non-classical Damping

The proposed method in this article can be used for systems with non-classical damping. For this purpose a two **DOF** system with non-classical damping subjected to a harmonic dynamic load is considered here. Forced vibration equation of the system is as follows:

$$M\ddot{u}+C\dot{u}+Ku=p(t),\qquad \mathrm{u}(0)=\begin{bmatrix}0\\0\end{bmatrix},\qquad \dot{u}(0)=\begin{bmatrix}0\\0\end{bmatrix}. \tag{45}$$

Where $M=\begin{bmatrix}0.2 & 0\\ 0 & 0.1\end{bmatrix}$, $C=\begin{bmatrix}0.112 & -0.0896\\ -0.0896 & 0.0896\end{bmatrix}$, $K=\begin{bmatrix}3 & -1\\ -1 & 1\end{bmatrix}$ and $p(t)=\begin{bmatrix}100\times\sin(0.8\ t)\\ 0\end{bmatrix}$.

Eq. (45) is equivalent to the following system of differential equations with initial values:

$$\begin{aligned} &0.2\ddot{u}_1+0.112\dot{u}_1-0.0896\dot{u}_2+3u_1-u_2=100\times\sin(0.8\ t)\ , \\ &0.1\ddot{u}_2-0.112\dot{u}_1+0.0896\dot{u}_2-u_1+u_2=0\ , \\ &u_1(0)=0,\quad u_2(0)=0,\quad \dot{u}_1(0)=0,\quad \dot{u}_2(0)=0\ . \end{aligned} \tag{46}$$

MDTM and Pade' approximant of order $\left[\frac{6}{6}\right]$ are applied on Eq. 46 and the analytical solutions are as follows:

$$u_1 = \exp(-0.0001t)\times(-0.5585\times\cos(0.8002t)+55.4342\times\sin(0.8002t)) + \exp(-0.0924t)\times(-0.7925\times\cos(2.2441t)-13.9605\times\sin(2.2441t)) + \exp(-0.6354t)\times(1.3510\times\cos(4.4063t)-2.7785\times\sin(4.4063)),$$

$$u_2 = \exp(-0.6354t)\times(-0.7303\times\cos(4.4063t)+3.1658\times\sin(4.4063t)) + \exp(-0.0930t)\times(1.6411\times\cos(2.2442t)-27.4640\times\sin(2.2442t)) + \exp(0.0004t)\times(-0.9108\times\cos(0.8005t)+59.1838\times\sin(0.8005t)). \quad (47)$$

In Figure (8), the diagrams of the functions $u_1(t)$ and $u_2(t)$ which result from **MDTM** and **RKF45** method are plotted. Comparison of the plotting curves shows remarkable accuracy of **MDTM**.

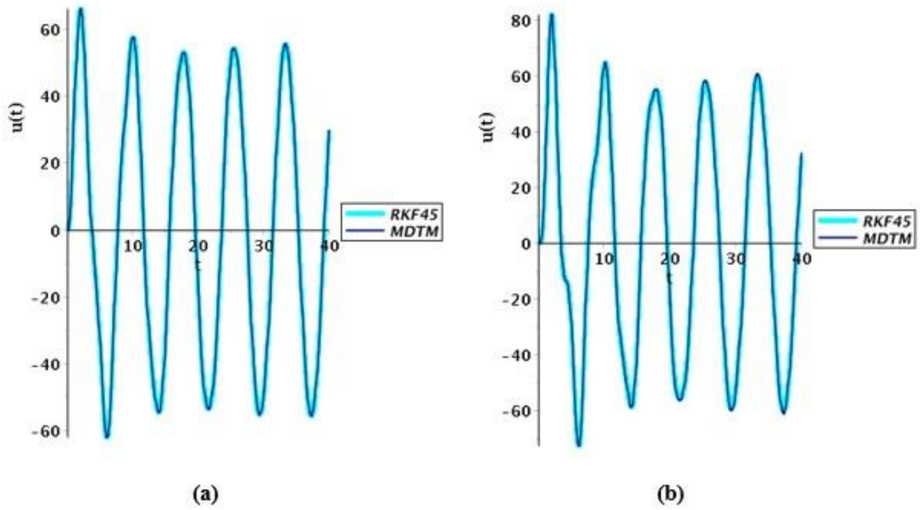

Figure 8. Response of the damped two DOF system a- u1(t), b- u2(t)

6.6. Free Vibration of a Damped Three DOF System

A three **DOF** system is considered in this example. Free vibration equation of the system is as follows:

$$M\ddot{u} + C\dot{u} + Ku = 0, \qquad \mathrm{u}(0) = \begin{bmatrix} 1 \\ 2/3 \\ 1/3 \end{bmatrix}, \qquad \dot{u}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (48)$$

Where $M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 2.583 & -0.471 & 0 \\ -0.471 & 3.526 & -0.943 \\ 0 & -0.943 & 4.94 \end{bmatrix}$, $K = \begin{bmatrix} 600 & -600 & 0 \\ -600 & 1800 & -1200 \\ 0 & -1200 & 3600 \end{bmatrix}$.

Eq. (48) is equivalent to the following system of differential equations with initial values:

$$\begin{aligned} &2\ddot{u}_1 + 2.583\dot{u}_1 - 0.471\dot{u}_2 + 600u_1 - 600u_2 = 0, \\ &2\ddot{u}_2 - 0.471\dot{u}_1 + 3.526\dot{u}_2 - 0.943\dot{u}_3 - 600u_1 + 1800u_2 - 1200u_3 = 0, \\ &2\ddot{u}_3 - 0.943\dot{u}_2 + 4.94\dot{u}_3 - 1200u_2 + 3600u_3 = 0, \\ &u_1(0) = 1,\ u_2(0) = \frac{2}{3},\ u_3(0) = \frac{1}{3},\ \dot{u}_1(0) = 0,\ \dot{u}_2(0) = 0,\ \dot{u}_3(0) = 0. \end{aligned} \quad (49)$$

MDTM and Pade' approximant of order $\left[\frac{10}{10}\right]$ are applied on Eq. 49 and its analytical solutions are as follows:

$$\begin{aligned} u_1 &= 1.3038\times10^{-7} + \exp(-1.3569t)\times(0.0583\times\cos(45.9170t) + 0.0017\times\sin(45.9170t)) \\ &+ \exp(-0.8246t)\times(0.0640\times\cos(27.4474t) + 0.0019\times\sin(27.4474t)) \\ &+ \exp(-0.5807t)\times(0.2107\times\cos(11.6366t) + 0.0105\times\sin(11.6366t))\ , \\[1em] u_2 &= \exp(-1.3569t)\times(0.005\times\cos(45.917t) + 0.0001\times\sin(45.917t)) \\ &+ \exp(-0.8246t)\times(-0.0738\times\cos(27.4474t) - 0.0022\times\sin(27.4474t)) \\ &+ \exp(-0.5807t)\times(1.0688\times\cos(11.6366t) + 0.0534\times\sin(11.6366t))\ , \\ &+ 3.6\times10^{-10}\times\exp(-26.5907t) \\[1em] u_3 &= -1.3748\times10^{-7} + \exp(-1.3569t)\times(-0.0301\times\cos(45.9170t) - 0.0009\times\sin(45.9170t)) \\ &+ \exp(-0.8246t)\times(0.1117\times\cos(27.4474t) + 0.0034\times\sin(27.4474t)) \\ &+ \exp(-0.5807t)\times(0.5852\times\cos(11.6366t) + 0.0292\times\sin(11.6366t))\ . \end{aligned} \tag{50}$$

In Figure (9), the diagrams of functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ which result from **MDTM** and **RKF45** method are plotted. Although this problem has more **DOF** with compare to the previous examples, but again comparison of the curves shows good accuracy of **MDTM**.

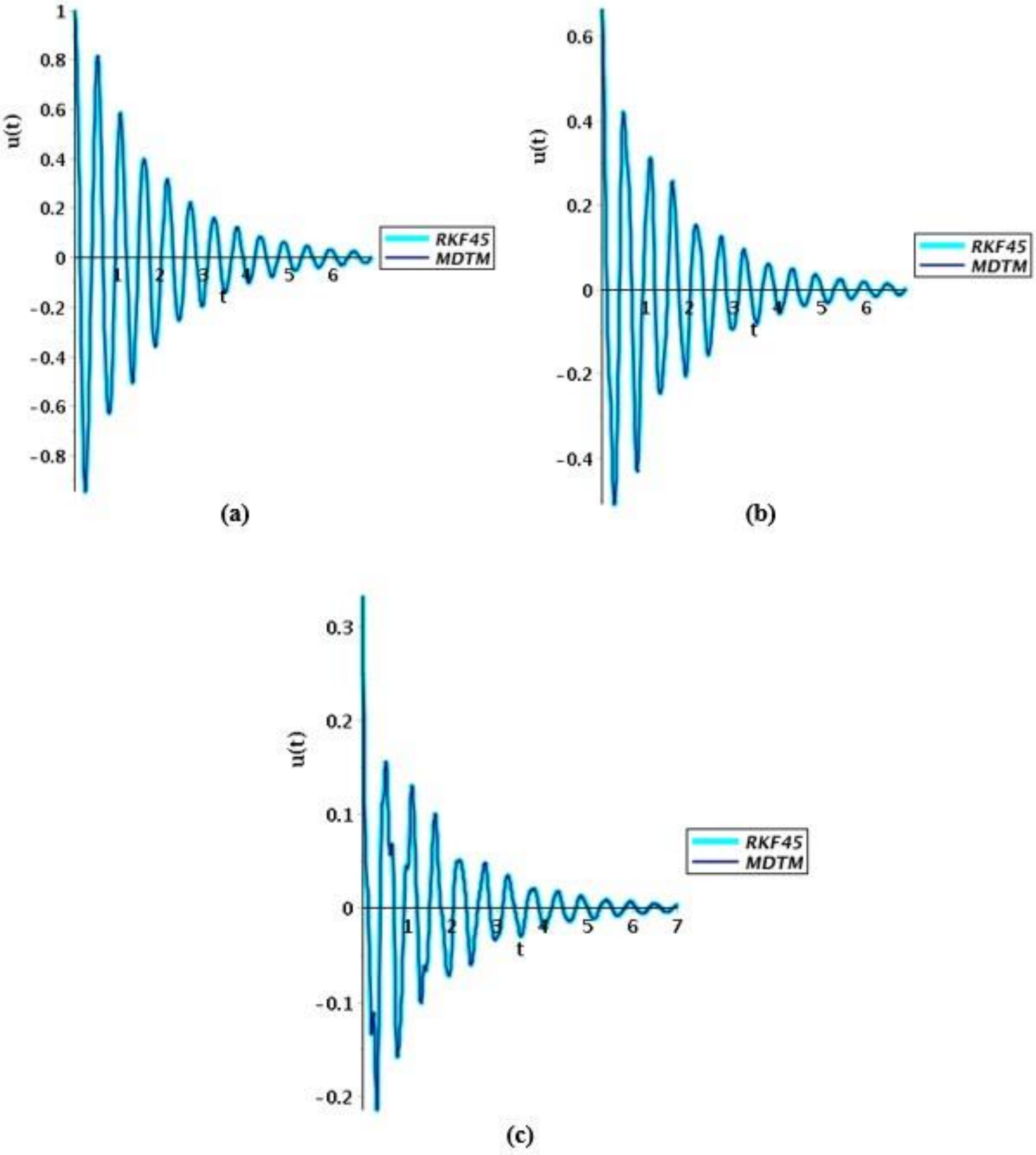

Figure 9. Response of the damped two DOF system a- u1(t), b- u2(t), c- u3(t)

6.7. Free vibration of an un-damped five DOF system

The last example that is considered in this article is free vibration of an un-damped five **DOF** system. Free vibration equation of the system is as follows:

$$M\ddot{u} + Ku = 0, \qquad \mathrm{u}(0) = \begin{bmatrix} 1/5 \\ 2/5 \\ 3/5 \\ 4/5 \\ 1 \end{bmatrix}, \qquad \dot{u}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{51}$$

Where $M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$.

The above equation can be written in the following equivalent form:

$$\begin{aligned}
&\ddot{u}_1 + 2\,u_1 - u_2 = 0\ , \\
&\ddot{u}_2 - \ u_1 + 2u_2 - u_3 = 0\ , \\
&\ddot{u}_3 - \ u_2 + 2\,u_3 - \ u_4 = 0\ , \\
&\ddot{u}_4 - \ u_3 + 2\,u_4 - \ u_5 = 0\ , \\
&\ddot{u}_5 - u_4 + \ u_5 = 0\ , \\
&u_1(0) = \frac{1}{5},\ \ u_2(0) = \frac{2}{5},\ \ u_3(0) = \frac{3}{5}\ ,\ \ u_4(0) = \frac{4}{5},\ \ u_5(0) = 1\ , \\
&\dot{u}_1(0) = 0,\ \ \dot{u}_2(0) = 0,\ \dot{u}_3(0) = 0\ ,\ \dot{u}_4(0) = 0,\ \dot{u}_5(0) = 0.
\end{aligned} \tag{52}$$

MDTM and Pade' approximant of order $\left[\frac{10}{10}\right]$ are applied on Eq. (52). In Figure (10), the diagrams of $u_1(t)$, $u_2(t)$, $u_3(t)$, $u_4(t)$ and $u_5(t)$ which result from **MDTM** and **RKF45** method are plotted. Comparison of the plotting curves shows that the response of the whole five degrees if freedom obtained by **MDTM** has good accuracy.

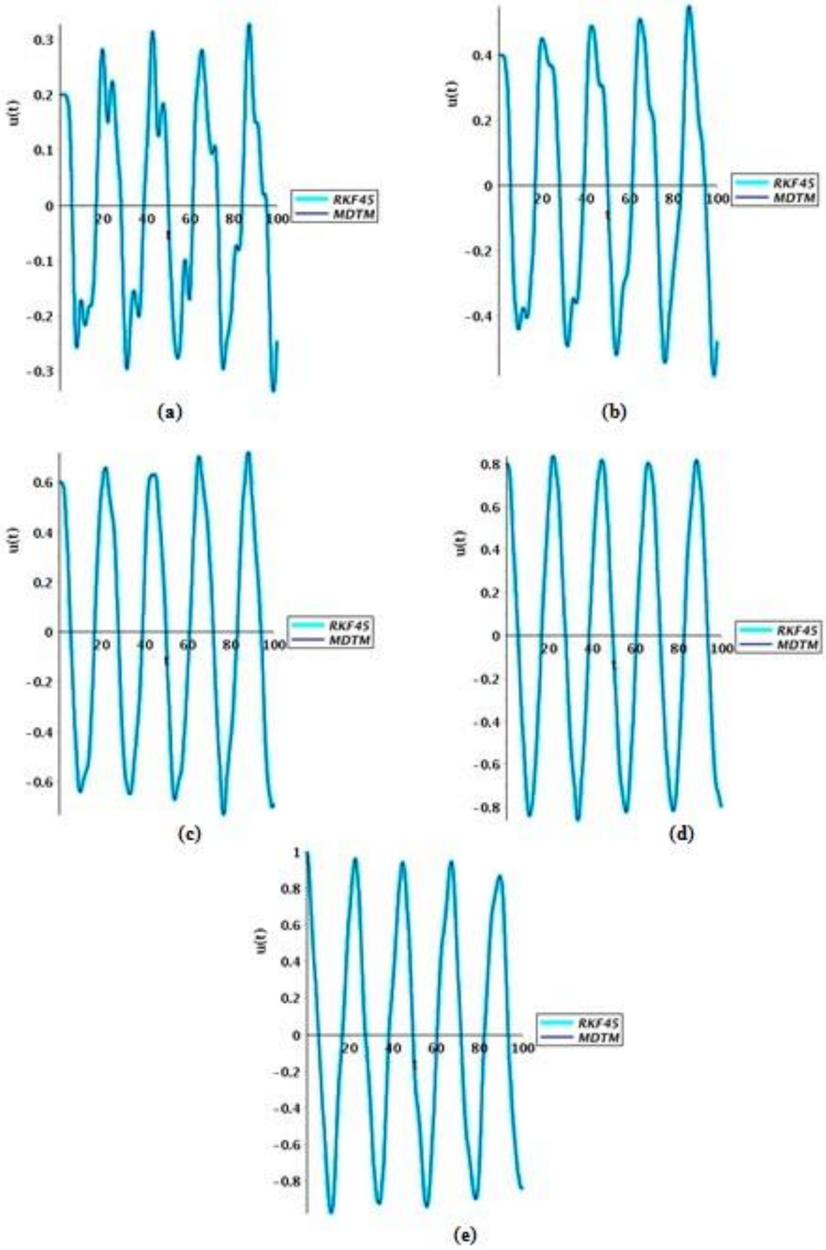

Figure 10: Response of the un-damped five DOF system a- u1(t), b- u2(t), c- u3(t), d- u4(t) e- u5(t).

7. Conclusions

In this article, the **MDTM** as a recursive semi-analytical method which is a hybrid of **DTM**, Pade' approximant and Laplace transformation is generalized for solving system of differential equations. Then it is applied on vibration equation of linear **MDOF** systems. Finally, a series of examples which include free and forced vibration of some structural systems with different number of degrees of freedom are solved by **MDTM** and the following conclusions are obtained:

1) The results which obtained by **DTM** don't have acceptable accuracy in large time intervals, so this method is not appropriate for solving the vibration equations of oscillatory systems. Therefore, using **MDTM** is inevitable.
2) The results which obtained by **MDTM** have acceptable accuracy. Also, low computational cost and simplicity are some superiorities of **MDTM** with respect to the others methods which are commonly used for solving the vibration equations of **MDOF** systems.
3) The method which is described in this article is usable for solving vibration equations of **MDOF** systems with classical and non-classical damping.
4) In some cases such as vibration of a single degree of freedom system, the solution which obtained by **MDTM** is the exact solution.

8. References

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