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Reliability Design of a New Masonry Bridge: An Approach Based on RBDO and Rigid Block Analysis

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Abstract

The objective of this study is to establish a reliability-based design framework for new masonry arch bridges, providing a rational alternative to the empirical rules that traditionally governed their construction. The proposed methodology integrates Reliability-Based Design Optimization (RBDO) with the rigid block limit analysis method to optimize key geometric parameters under uncertain loading conditions. The probabilistic formulation incorporates the variability of geometric and load parameters, which are identified as the dominant sources of uncertainty during the design phase of new masonry bridges. Two RBDO strategies are employed: the Performance Measure Approach (PMA) and Sequential Optimization with Reliability Assessment (SORA), both coupled with a linear programming formulation of equilibrium and yield constraints. The approach is applied to the reconstruction of the historical Dar El Makina bridge in Fes, Morocco, to determine the optimal geometric configuration that satisfies target reliability requirements. The results indicate that the optimized design achieves a 27% reduction in arch thickness and a 13% increase in rise compared to the existing structure, leading to a safer and more material-efficient configuration. Compared with classical empirical formulas, the proposed approach provides a rational and quantitative basis for the design of masonry bridges, combining structural safety, material efficiency, and heritage preservation.

Keywords: Masonry Arch Bridge; Reliability-Based Design Optimization; Rigid Block Analysis; Structural Reliability.

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1. Introduction

Masonry arch bridges represent one of the most durable and aesthetically remarkable forms of civil engineering heritage. Their ability to withstand centuries of service, including modern traffic loads, attests to the ingenuity of historical builders and the intrinsic robustness of masonry construction. Historically, these structures were designed using empirical rules derived from experience rather than analytical or probabilistic models. While these empirical approaches have proven reliable for existing bridges, their direct application to the design of new masonry bridges remains limited, as modern engineering practice requires a rational framework that quantifies both safety and material efficiency.

To address this need, reliability-based design has emerged as a powerful approach to account for uncertainty in engineering systems. Within this context, Reliability-Based Design Optimization (RBDO) combines optimization techniques with probabilistic reliability assessment to achieve designs that meet prescribed safety targets while minimizing material use or cost. Over the past decade, significant methodological advances have improved its robustness and computational efficiency through adaptive surrogate modeling and refined search algorithms [1-3]. Recent

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applications to bridge engineering—such as deck-shape optimization in long-span suspension bridges [4] and life-cycle cost-based RBDO of coastal bridge piers [5]—have demonstrated the potential of probabilistic optimization to enhance both safety and performance in civil infrastructure. However, to the best of the authors' knowledge, RBDO specifically applied to the design of new masonry bridges remains largely unexplored in the literature.

To enable such reliability-based optimization, an appropriate mechanical model capable of representing the discontinuous behavior of masonry is required. Numerous models have been proposed for this purpose, ranging from detailed finite element formulations to simplified macro-element or rigid-block approaches. Among these, the rigid-block limit analysis method, originally introduced by Livesley (1978) [6] and later refined by Gilbert (2007) [7], offers an appealing balance between physical realism and computational efficiency. This model assesses the structural safety of a stack of rigid blocks, connected by mortar joints, whose failure is governed by Mohr-Coulomb behavior. Any sliding or excessive rotation, often caused by external live loads, leads to the formation of localized loss of contact at a joint, known as a hinge. The collapse of the masonry bridge occurs when enough hinges develop, transforming the structure into a mechanism.

Recently, a renewed interest in modern stone arch construction and mechanically optimized forms has emerged. Nodargi & Bisegna (2020) [8] revisited the classical thrust-line theory and proposed an optimization framework capable of identifying the geometrically optimal profiles of masonry arches under self-weight, providing a rigorous analytical basis for contemporary design. Todisco et al. (2025) [9] investigated the design, construction, and testing of post-tensioned stone arch footbridges, demonstrating the feasibility of combining traditional masonry with controlled prestressing. Similarly, Adiels (2023) [10] introduced the hydrostatic shell concept, in which the arch geometry follows the theoretical lines of thrust to achieve uniform compressive stress distribution. These studies highlight how contemporary analytical tools can renew the design of masonry structures. Nevertheless, as noted by Orfeo (2023) [11], such realizations remain rare, and comprehensive reliability-based design methodologies tailored to new masonry bridges are still missing.

Building on these developments, the present study proposes a reliability-based design framework for masonry arch bridges that integrates RBDO with the rigid-block limit analysis method. The framework focuses on optimizing the key geometric parameters—namely, the arch thickness and rise—while accounting for the uncertainty associated with design and load variables. This focus is particularly relevant for the construction of new masonry bridges, where material properties can be adequately controlled through specifications and quality management. The methodology is applied to the case of the historical Dar El Makina bridge in Fes, Morocco, chosen for its cultural significance and representative geometry. By combining the mechanical rigor of limit analysis with the probabilistic consistency of Reliability-Based Design Optimization (RBDO), this work establishes a rational and systematic framework for the modern design of masonry bridges. The proposed approach provides a scientific alternative to the traditional empirical methodologies historically adopted for this class of structures, ensuring both structural safety and material efficiency through reliability-driven optimization.

In this framework, two complementary RBDO strategies are employed and coupled with the rigid-block mechanical model. The first, known as the Performance Measure Approach (PMA), determines the most probable failure point for a prescribed target reliability index using an inverse First-Order Reliability Method (FORM). The second, the Sequential Optimization and Reliability Assessment (SORA) method, decomposes the RBDO problem into two sequentially solved sub-problems: a deterministic optimization stage and a reliability evaluation stage. This decoupling significantly enhances convergence stability and reduces computational cost while maintaining accuracy. The resulting framework thus bridges the gap between empirical design traditions and contemporary reliability-based standards, offering a robust pathway for the rational design of new masonry bridges. This work complements the authors' previous study [12], which addressed the reliability assessment of existing masonry bridges considering both material and geometric uncertainties.

2. Structural Reliability

In the context of constructing complex structures, optimal design is essential to strike a balance between performance and cost. The goal of structural design optimization is to minimize costs while ensuring that the structure satisfies safety requirements. However, when uncertainties impact design parameters or loading conditions, a structure designed according to deterministic principles may operate outside its nominal specifications. Therefore, it is crucial to account for uncertainties in the early design stages. This is achieved through a reliability study, which aims to estimate the risk of failure by considering factors such as geometry, materials, and loads [13] as random variables. The reliability problem can be characterized by a vector of random variables $\boldsymbol{X}$ and the subset or domain Ω of realizations. This domain can be separated into two subdomains, Ω_f, which corresponds to a failure domain, and Ω_s, which relates to a safety domain. Failure events are defined in respect to a limit state function $g(\boldsymbol{X})$. The probability of failure P_f is defined by the integral involving the function $f_{\boldsymbol{X}}(\boldsymbol{x})$, which is the probability density function (PDF) of the vector $\boldsymbol{X}$ defined by Equation 1:

$$P_f = P(\boldsymbol{x} \in \{\Omega_f \equiv g(\boldsymbol{x}) \leq 0\}) = \int_{g(\boldsymbol{x})\leq 0} f_{\boldsymbol{X}}(\boldsymbol{x}) dx \tag{1}$$

Several approaches for estimating this probability exist in the literature [13-15], mostly comprising approximation and simulation methods. Simulation methods are generally based on sampling and creating random samples that allow for the calculation of the previous integral. Moreover, Approximation methods provide a more cost-effective approach, offering a simplified means to evaluate the limit state function. Depending on the degree of complexity and non-linearity of the limit state function, one can employ first-order techniques (FORM) with linear approximation or second-order methods (SORM) with quadratic approximation. A fundamental step in the execution of the FORM approach is the iso-probabilistic transformation, which entails migrating to a standard space where random variables are represented by standardized normal variables (Figure 1 and Equation 2).

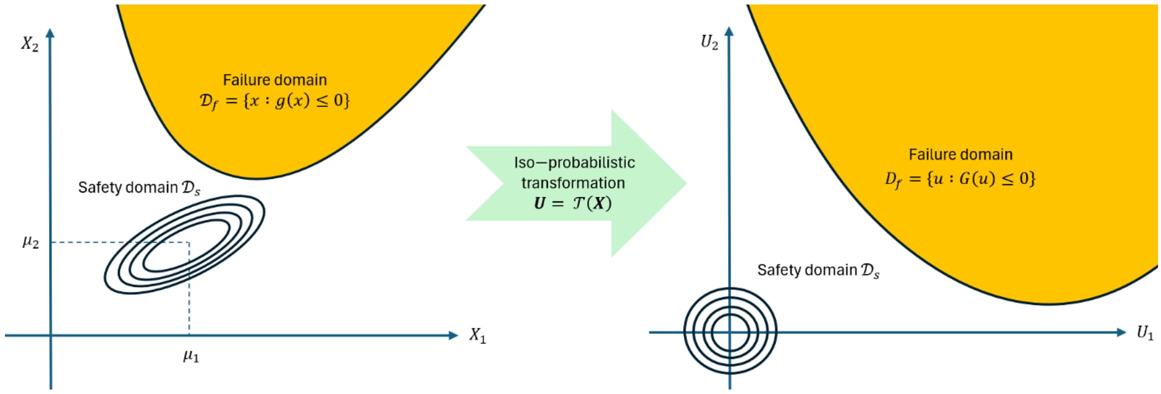

Figure 1. ISO-probabilistic transformation

$$U = \mathcal{T}(X) \; ; U \backsim \mathcal{N}(0, I) \tag{2}$$

The FORM approach relies on linearizing the limit state function at the approximate location to the origin of the probabilistic space, known as the design point [13] (Figure 2).

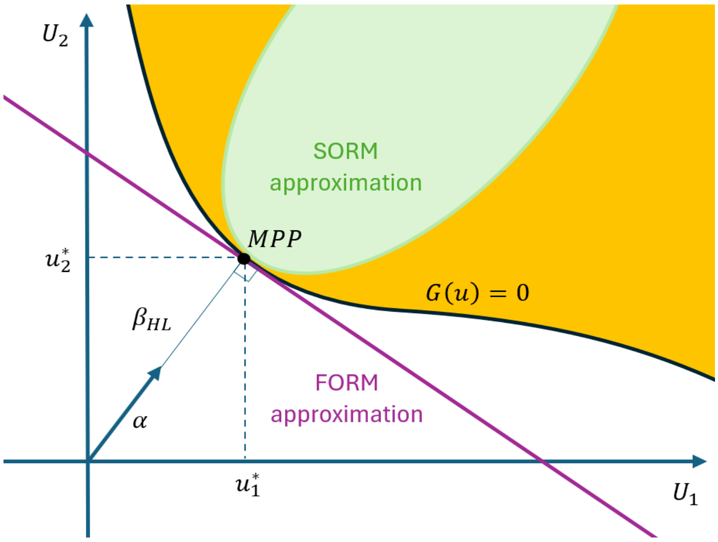

Figure 2. FORM & SORM approximation

To accurately determine low values for the probability of failure (approximately 10^{-5}), a complex sampling technique based on the Monte Carlo approach would be required. In general, the reliability index β_{HL} introduced by Hasofer and Lind is more precise in expressing the probability of failure as in Equation 3 [13]:

$$P_f = \Phi(-\beta_{HL}) \; ; \text{with } \beta_{HL} = \min_{G(u) \leq 0} \left(\sqrt{\{u\}^t \{u\}}\right) \tag{3}$$

where Φ is the cumulative gaussian probability cumulative distribution function (CDF). This fundamental relation of the FORM algorithm demonstrates that the overall safety of the structure increases as the reliability index β increases. An iterative search for the solution is conducted utilizing optimization techniques, such as the projected gradient method. The solution of the optimization problem that aims to the calculation of reliability index is known as the Most Probable Point (MPP); it is the point where the failure has the highest probability. The FORM algorithm as well as inverse FORM algorithm are used in the RBDO methods employed in this paper.

3. Reliability-Based-Design-Optimization (RBDO)

To directly address uncertainties in structural design, Reliability-Based design optimization (RBDO) is often considered as an alternative to Deterministic Design Optimization (DDO). RBDO aims to achieve a balance between cost-effectiveness and safety by accounting for the inherent uncertainties in the structural model. Unlike the traditional DDO approach, where partial safety factors γ are applied in a fixed manner to ensure design robustness, RBDO directly incorporates failure probabilities and random variations of parameters into the optimization process [16] (Figure 3). This ensures that the design meets safety requirements while optimizing performance, avoiding the systematic over-sizing of structures often associated with partial safety factors.

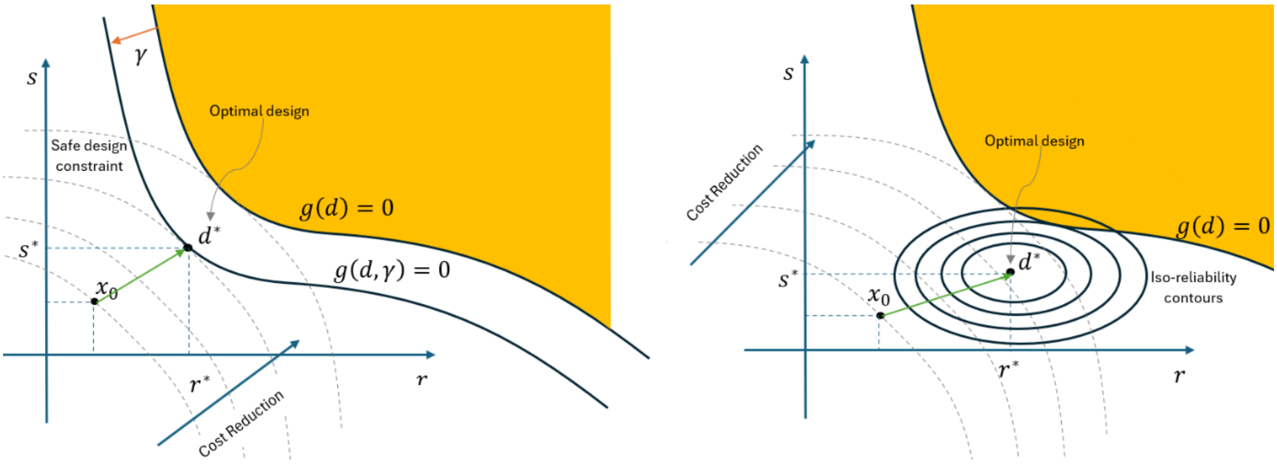

Figure 3. Comparison between DDO (left) and RBDO (right)

Various applications of Reliability-Based Design Optimization (RBDO) have been explored for different structural systems in the literature. Aoues & Chateauneuf [17] applied RBDO to achieve an optimal balance between system reliability and component performance in reinforced concrete frames. Gholizadeh & Mohammadi [18] addressed RBDO in the seismic context for steel frames, while incorporating soil-structure interaction as a metamodel in the RBDO of reinforced concrete structures. More recently, reliability optimization has been used to optimize the design of complex structural systems. Siacara et al. [19] focused on classical failure modes of a concrete dam, while Yu et al. [20] integrated topology and shape optimization into the RBDO framework for designing marine structures.

The RBDO solution is primarily obtained by simultaneously conducting a reliability analysis and solving an optimization problem. However, this coupling introduces challenges that make the appealing concept of RBDO more difficult to implement. Various methods for solving the RBDO problem have been proposed in the literature, and they can be classified based on the reliability analysis technique used [16]:

- Approximation approaches: These rely on conventional approximation methods within structural reliability theory, such as the First-Order Reliability Method (FORM), to perform reliability analysis. Traditional RBDO methods fall into this category.
- Simulation approaches: These are based on Monte Carlo simulation methods, which, while more computationally expensive, offer higher accuracy.

Chateauneuf (2008) [21] identified several approximation methods commonly used in literature, including the Reliability Index Approach (RIA), the Performance Measure Approach (PMA) and Sequential Optimization with Reliability Assessment (SORA). These diverse methodologies are integrated into the UQLAB computational framework [22], which is employed in this paper for the Reliability-Based Design of a masonry bridge.

3.1. General Formulation of a RBDO Problem

In the general framework, the Reliability-Based Design approach can be formulated as the process of finding the optimal value of a cost function, while ensuring that multiple constraints are met. These constraints account for the various uncertainties associated with the problem in Equation 4:

$$d^* = \text{argmin}_{d \in \mathbb{D}}\, c(d) \;\; s.t.: \begin{cases} f_j(d) \le 0, & \{j = 1, \dots, s\} \\ P(g_k(X(d), Z) \le 0) \le \bar{P}_{f_k}, & \{k = 1, \dots, n\} \end{cases} \tag{4}$$

The objective of the study is to minimize the cost function $c(d)$ with respect to the design parameters or variables d within the design space $\mathbb{D} \subset \mathbb{R}^{M_d}$. These variables are subject to deterministic constraints, known as "soft constraints," represented by $f_j(d)$, which limit the search space for the design variables. Additionally, the reliability aspect is incorporated through "hard constraints" which impose strong restrictions based on the probabilistic nature of the design variables $X(d)$ as well as environmental variables Z.

These constraints involve limit state functions computed according to the chosen mechanical model and can be represented in two ways: either directly using the probability of failure P_f or alternatively, by employing the reliability index β. The second option, which is often preferred to avoid numerical instabilities associated with calculating P_f, is formulated as the RBDO Equation 5:

$$d^* = \text{argmin}_{d\in\mathbb{D}}\, c(d)\;\; s.t.: \begin{cases} f_j(d) \le 0, & \{j = 1, \dots, s\} \\ \bar{\beta}_k - \beta_k(X(d), Z) \le 0, & \{k = 1, \dots, n\} \end{cases} \tag{5}$$

3.2. Methods for RBDO Analysis

In RBDO, the reliability problem is addressed by solving the optimization problem, implemented in the space of the random variables. Traditional RBDO necessitates a two-loop iteration process, wherein reliability analysis is conducted in the inner loop for every modification in the design parameters, to assess the reliability constraints. The computational complexity of this method is significantly increased by the multiplication of the number of iterations in both optimization and reliability problems, which include many mechanical model evaluations. A range of approaches for tackling an RBDO problem are used to find a solution for the RBDO problem, including:

- Two-level methods: the two levels correspond to two nested loops, with the outer loop exploring the design space while the inner loop estimates the likelihood of failure. Notably, the RIA and PMA approaches belong to this group.
- Methods referred to as single level: for which a single loop is employed, but on an iterative computation where the problem is changed into a deterministic problem. This conversion can be carried out based on the Karush-Kuhn-Tucker optimality requirements, for example, to replace reliability constraints with deterministic constraints [16].
- Decoupled approaches: these methods allow for the decoupling of the optimization problem—which is tackled sequentially in a deterministic framework—from the reliability calculation problem; the SORA algorithm is part of this group.

Simulation-based RBDO methods such as Monte Carlo or Latin Hypercube sampling may be excluded because of their high computational cost relative to the limited number of random variables. Each model evaluation involves solving the mechanical problem, which makes repeated sampling impractical. Instead, gradient-based reliability methods introduced above have proven accuracy and efficiency in low- to moderate-dimensional probabilistic spaces. For this study, the two-level methods and decoupled approaches are applied and compared:

3.3. RIA Algorithm (Reliability Index Approach)

This is the most widely used method among two-level approaches based on the FORM formulation for the inner loop and involving cost reduction under reliability index constraints. The two nested optimization loops operate concurrently, meaning that for each new set of design parameters, a new design point is identified through reliability analysis (Equation 6):

$$\beta_k = \|u_k^*\|$$
$$\text{Whilst: } u_k^* = argmin_{u\in\mathbb{R}^M}\{\|u\|, G_k(u) \le 0\}\; et\; G_k(u) = g_k(\mathcal{T}^{-1}(u)) \tag{6}$$

Optimization algorithms such as Hasofer and Lind–Rackwitz and Fiessler (HL–RF) or its improved version (iHL–RF) are used to solve Equation 6 and find the design point. The successive adjustments of the optimal point and the design point in this method lead to slow convergence and erratic, zigzagging behavior. It is widely acknowledged that the Reliability Index Approach (RIA) often demonstrates slow convergence and, in some cases, may even fail to converge altogether [23].

3.4. PMA Algorithm (Performance Measure Approach)

The PMA is based on an inverse FORM analysis to obtain the Minimum Performance Target Point (MPTP) for a fixed target reliability index $\bar{\beta}$. This point is defined as in Equation 7:

$$u_{MPTP} = argmin_{u\in\mathbb{R}^M}\{G_k(u), \|u\| = \bar{\beta}\} \tag{7}$$

PMA approach serves as an alternative to RIA, relying on the definition of a target performance level instead of as a simpler constraint compared to RIA. While RIA exhibits zigzagging behavior, the PMA approach first directs the optimization toward the hypersphere with a radius equal to the target reliability index [22]. Subsequent iterations are then performed on this hypersphere. This process results in faster and more stable convergence in the case of PMA.

General optimization algorithms can be utilized in inverse FORM; while the Advanced Mean Value (AMV) method is well-suited due to its simplicity and efficiency, it tends to exhibit instability and inefficiency when dealing with concave performance measures. To address these issues, the Hybrid Mean Value (HMV) method has been introduced, offering improved numerical efficiency and stability [23].

3.5. SORA Algorithm (Sequential Optimization with Reliability Assessment)

The Sequential Optimization with Reliability Assessment (SORA) approach seeks to reduce the RBDO issue into numerous deterministic optimization cycles to be solved successively. It is based on a single loop with a sequence of deterministic optimization and reliability analyses. The design point is obtained in each cycle by an inverse FORM technique and is utilized to move the design variables in the following cycle. During each cycle i, the formulation of the RBDO problem is approached by Equation 8:

$$d^* = \text{argmin}_{d\in\mathbb{D}}\, c(d)\ avec: \begin{cases} f_j(d) \leq 0, & \{j = 1, \dots, s\} \\ g(d - \sigma_k{}^{(i)}, z_k{}^{(i-1)}) \geq 0 & \{k = 1, \dots, n\} \end{cases} \tag{8}$$

where, $\sigma_k{}^{(i)} = d^{(i-1)} - x_{MPTP_k}{}^{(i-1)}$ is the shift at cycle (i).

The appeal of this approach lies in its entirely deterministic formulation, allowing for the use of straightforward optimization algorithms with precision like Sequential Quadratic Programming (SQP). However, the SORA approach typically requires a greater number of mechanical model evaluations, which can reduce its overall efficiency [24].

4. Methodology

4.1. Mechanical Model: Rigid Block Analysis

The method of rigid block analysis entails considering the structure of the masonry arch as a series of stone blocks that have infinite compressive strength and negligible tensile strength, a multiplicative factor of an imposed unit load is being sought. A 2D version of the method is presented below, which is easier to couple with RBDO approaches.

The equations governing this technique have been published in various reference publications and form the basis of the RING software [25]. According to these equations, the search for the collapse multiplier fundamentally includes using the classical lower bound limit analysis. An optimization problem is formulated within equilibrium equations and yield conditions, either through plastic hinges or by sliding including a coefficient of friction. The bridge is composed of N blocks and $N + 1$ contact surface, is subject to dead loads due to self-weight F_D and live loads due to the moving axles of the truck F_L. The equilibrium is expressed as in Equation 9:

$$E * q_x = F_D + F_L \tag{9}$$

where $\boldsymbol{E}$ is the equilibrium matrix and the vector $\boldsymbol{q_x}$ gathers the contact forces (shear force, normal force, moment) (Figure 4) at the interfaces between blocks (Equation 10):

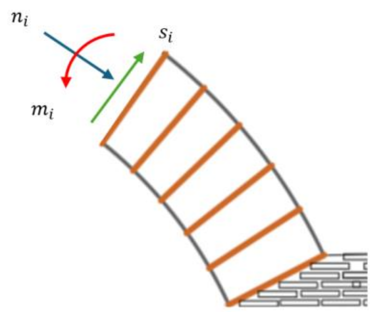

Figure 4. Contact forces at joints $\boldsymbol{q_x}$

$$q_x = \{(n_i; s_i; m_i); i = 1, N + 1\} \tag{10}$$

Subsequently, it is necessary to express the relationships between the different quantities forming the vector q_x, which are yield constraints that concern all $N + 1$ contact surfaces of the arch:

- No tension or rocking yield constraints (Equation 11):

$$|m_i| \leq 0.5 * n_i * e\ ; i = 1, N+1 \tag{11}$$

- No sliding yield constraints (Equation 12):

$$|s_i| \leq \tan(\phi) * n_i;\ ; i = 1, N+1 \tag{12}$$

where, e is the thickness of arch and $\tan(\phi)$ is the friction coefficient of the joint.

Classically, the aforementioned conditions 9, 11 and 12 are the basis for an optimization problem to be solved using a linear programming approach. The resolution yields the failure limit load ${F_L}^{lim}$, the contact force vector q_x, as well as the location of the contacts that have failed, thereby transforming the arch into a mechanism. The mechanism is obtained either via enough hinges or planes of sliding, in the example of two-span arch bridge, the 7 hinges mechanism is often observed (Figure 5).

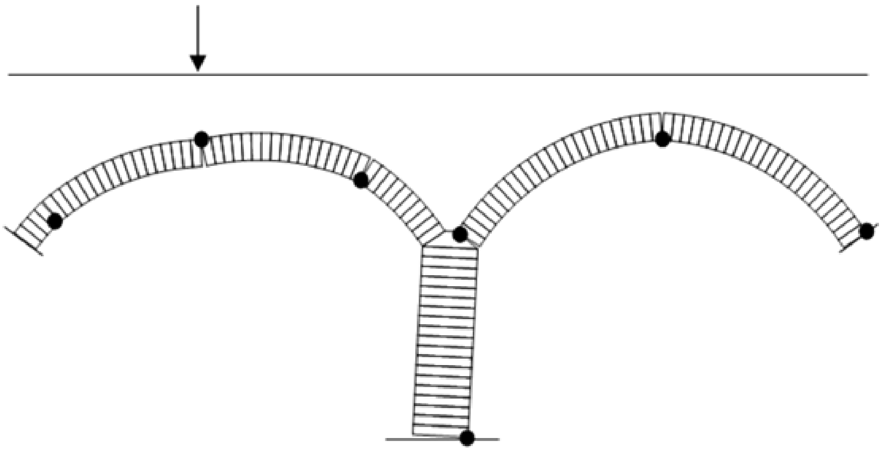

Figure 5. 7-hinges mechanism for two-span masonry arch bridge [7]

The rigid-block model is implemented in its simplified limit-analysis form, assuming infinite compressive strength and no tensile resistance for the masonry, consistent with Heyman's classical theory [26]. These assumptions eliminate the nonlinear material behavior and restrict the problem to geometric and equilibrium considerations, where failure occurs through hinge formation rather than crushing. Under these conditions, the limit-state function is piecewise linear and differentiable near the collapse mechanism, allowing the Hasofer–Lind First-Order Reliability Method (FORM) to provide an accurate and computationally efficient estimation of the reliability index generally sufficient in the RBDO framework [27].

4.2. Ancient Arch Bridge to Rebuilt-Optimization Issue

Traditionally, the determination of the geometric parameters of the arch was based on empirical rules, which first allowed for the calculation of the ratio of the rise h to the span l. For segmental arches, this ratio reached values between 1/6 and 1/9 with adequate angles at springing [28]. The arch thickness at crown e is a crucial parameter for the stability of the arch, as it plays a role in the mechanical equilibrium equations, as discussed in the following section. Several authors, from various periods since the 15th century, have proposed empirical formulas for calculating this thickness based on the span of the arch, its extrados radius in the case of a semicircular arch, its rise, and its composition (brick, stone, etc.). However, it is worth noting that these formulas yield disparate values and do not adequately account for the effects of external loads applied to the structure.

In the present framework, the random variables include the geometric parameters of the arch and the applied loads, which represent the dominant sources of uncertainty during the design stage of a new masonry bridge. Material properties are assumed to be controlled through construction specifications and quality management procedures. This assumption is consistent with the focus on new bridges, where material variability can be minimized.

The methodology proposed in this paper involves evaluating the geometric specifications of the bridge to achieve an optimal balance between safety requirements and material usage, considering the effects of external loads. In this study, two reliability-related parameters are considered: the thickness of the arch e and the rise of the bridge h. These parameters are design variables to be determined through RBDO analysis using the previously described mechanical model. The ratio of these two parameters reflects the segmental shape of the bridge and, consequently, its efficiency in terms of material consumption. Environmental variables are represented by the target live loads whose value $\overline{F_L}$ is treated as random variable while position $\zeta_{\overline{F_L}}$ with respect to the axis of the first arch is kept deterministic. Other geometric parameters of the arch and the mechanical properties of the masonry blocks are treated as deterministic to simplify the problem. In (Figure 6) main parameters that are involved in the RBDO analysis are presented.

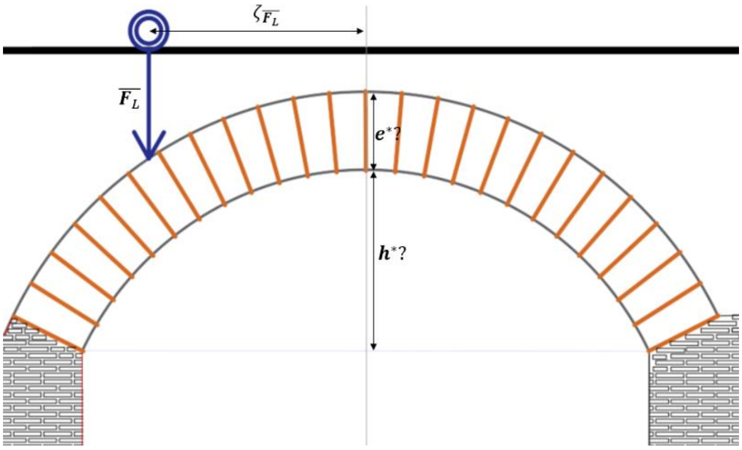

Figure 6. Geometrical parameters involved in RBDO

4.3. Formulation of the RBDO Problem

The Reliability-Based optimization problem will be based on the definition of reliability constraints in addition to the cost function that must be minimized. Other so-called soft constraints will also be considered to limit the search range for the two basic parameters e and h. These intervals can be established based on empirical knowledge enriched by the actual dimensions of the ancient bridge. The mechanical model that will be recommended to define the limit state is the rigid block analysis model described previously, simulating the failure of the bridge when one of yield constraints is violated.

The methodology proposed in this work enables the transformation of deterministic yield constraints into Reliability-Based constraints, with the cost function defined as the ratio of the thickness e to the rise h. This approach aims to maximize the slenderness of the new bridge while minimizing material usage. The parameters of the RBDO formulation are derived from a "reverse" limit analysis, which provides the contact forces within the arch structure based on geometrical and mechanical parameters, as well as the target live load $\overline{F_L}$ and its position $\zeta_{\overline{F_L}}$ relative to the axis of the first arch (Equation 13):

$$q_x = q_x\big(e; h; l; R; w; N; \gamma_m; \tan(\phi)\,; \overline{F_L}; \zeta_{\overline{F_L}}\big) \tag{13}$$

The mathematical formulation of RBDO problem is given as following (Equation 14):

$$d^* = \underset{d\in\mathbb{D}}{argmin}\; c\big(d = (e; h)\big) = argmin\left(\frac{e}{2h}\right)\; s.t.:$$
$$\begin{cases} m_i - 0.5 * n_i * e \leq 0, & \{i = 1, \dots, N+1\} \\ -m_i - 0.5 * n_i * e \leq 0 & \{i = 1, \dots, N+1\} \\ s_i - \tan(\phi) * n_i \leq 0 & \{i = 1, \dots, N+1\} \\ -s_i - \tan(\phi) * n_i \leq 0 & \{i = 1, \dots, N+1\} \\ \bar{\beta}_i - \beta_i((e; h), \overline{F_L}) \leq 0, & \{i = 1, \dots, N+1\} \end{cases} \tag{14}$$

The structural reliability evaluation of the studied arch must be conducted by comparing the calculated reliability index with the target reliability index $\bar{\beta}_i$, which is provided in several references in the literature. The target reliability index is generally provided considering the importance of the structure and the severity of the economic and social damages incurred because of its failure. If the consequences of the bridge failure are considered low, the probabilistic code of the JCSS [29] related to reliability design sets a minimum target reliability index of 3.10 for minor consequences of failure, while the Eurocode [30] proposes a value of 3.30 for the structure that belongs to a class of consequences CC1; both values correspond to the design of a new structure, the target value chosen for this paper is $\bar{\beta}_i = 3.20$.

The methodology of the study is summarized in the methodological workflow presented in Figure 7.

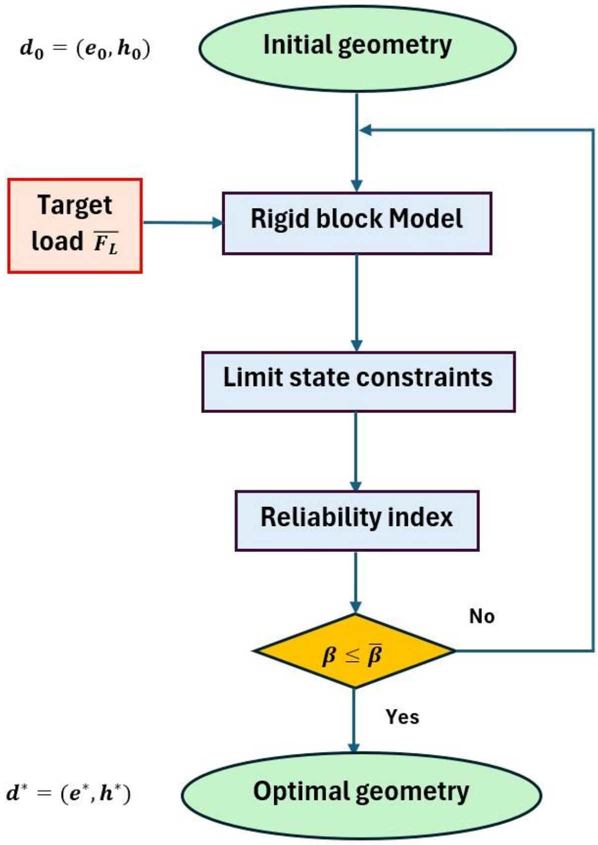

Figure 7. Methodological workflow

5. Case study: Ancient Bridge of Dar El Makina

5.1. Ancient Bridge Description

The ancient masonry bridge is part of the rehabilitation project of the Dar El Makina Garden in the old Medina of Fes in Morocco (Figure 8). The aim of the intervention involves the reconstruction of various monuments going back to the late 19th century. The project intends to reconstruct an ancient brick masonry bridge with the same architectural qualities as the old damaged one (Figure 9). The ancient bridge primarily functioned as a pedestrian bridge. It consisted of two semi-circular arches, built of brick masonry with a regular arrangement in a staggered pattern. The span of the bridge is of 4.67 meters and the arch thickness at the keystone is 0.60 meters. The arches rest on two abutments and a pier with a rise of 1.557 meters composed of rough stone masonry (Figure 10, Table 1).

Figure 8. Location of the study area: Dar El Makina Gardens – Fez – Morocco

Figure 9. Dar El Makina Ancient masonry bridge: in early 1920s (left); Nowadays (right)

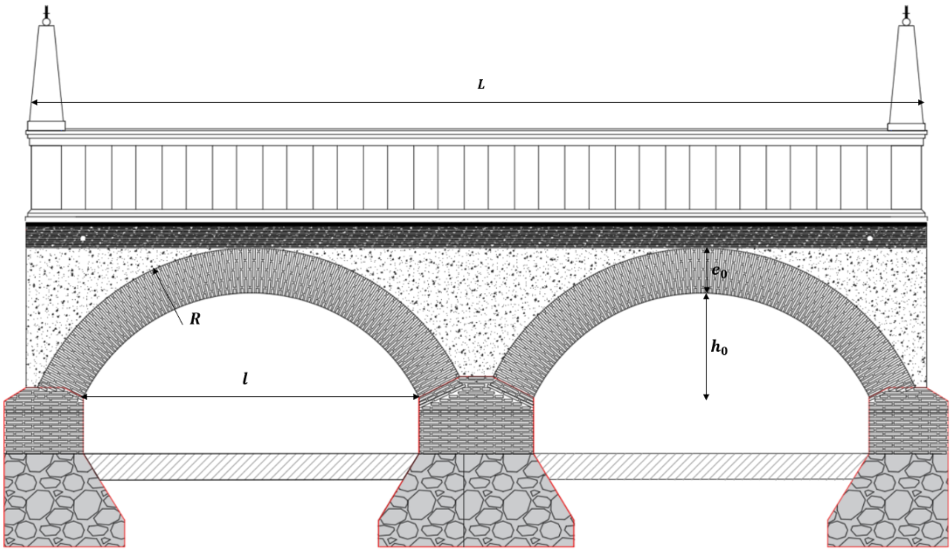

Figure 10. Main geometric parameters of Dar El Makina Ancient masonry bridge

Table 1. Characteristics of ancient bridge of Dar El Makina in Fes

Parameter*		Value
Total length to cross	L	11.98 m
Width of the bridge	w	4.00 m
Radius at extrados	R	3.09 m
Span of the bridge	l	4.67 m
Rise at the supports	h_0	1.575 m
Thickness at the keystone	e_0	0.60 m

The existing bridge occasionally supports the passage of trucks, despite not being originally designed for this purpose. The role of the proposed Reliability-Based design approach is to ensure that the new bridge can support live loads through an optimized design. The approach initially used the dimensions of the ancient bridge - based on empirical rules - as a starting point. The proposed design approach requires accounting for the live loads generated by the passage of vehicles or trucks, while preserving the geometrical characteristics of the existing bridge and incorporating the mechanical properties outlined in Table 2.

Table 2. Additional characteristics of the new bridge of Dar El Makina in Fes

Parameter		Value
Number of blocks per span	N	25
Density of masonry blocks	γ_m	20 kN/m^3
Friction coefficient	$\tan(\phi)$	0.60

* Index 0 is for initial values of design parameters to be modified in RBDO

To compare the results obtained through the RBDO approach with those provided by empirical rules, several formulas from periods close to the construction date of the studied bridge were implemented for calculating the arch thickness. These approaches differ not only in their limitations regarding the geometry of the arch but also in the nature of the parameters used in the calculations. A notable disparity in the obtained values can be observed, as shown in Table 3.

Table 3. Thickness of the arch obtained via empirical rules for studied bridge

Empirical rule [28]		Value for e
Rankine	$e = 0.191 * R^{0.5}$	0.334 m
Résal	$e = 0.15 + 0.20 * \frac{l}{2*\sqrt{h}}$	0.501 m
Kaven	$e = 0.25 + l * (0.025 + 0.00333 * \frac{l}{h})$	0.408 m
German & Russian Engineers	$e = 0.43 + 0.05 * l$	0.664 m

5.2. Choice of Load Distribution

The uncertainty of the live loads imposed on the bridge is represented through an appropriate probability density function (PDF). Empirical measurements can be performed to determine the frequency, load intensity, and load distribution, which are necessary for de reliability analysis.

Several authors have extensively used the data gathered by road network to identify appropriate loading models for their reliability studies on masonry bridges [31, 32]. The selection of the loading type is generally based on concentrated loads, while considering uniform loads for the reliability analysis of failure can potentially result in an underestimation of the failure loads [33].

The mechanical model adopted in this paper is not particularly concerned with the transverse distribution of the load as a 2D model is considered. A Gaussian distribution with a mean value for the axle load of 120kN and a coefficient of variation (CoV) of 15% is adopted as suggested by Matos et al. [34] (Table 4).

Table 4. Probability characteristics of target live load $\overline{F_L}$

Marginal type	Gumbel
$\mu(\overline{F_L})$	120 kN
CoV$(\overline{F_L})$	15%
Position of the axle w.r.t. 1st arch axis: $\zeta_{\overline{F_L}}$	-2.23 m

The selection of the load position is critical in the design of a masonry arch, as it determines the location of potential hinges that may form during failure, as well as the maximum load the structure can bear. Traditionally, the quarter-span of the arch are the most vulnerable areas, often assigned the lowest load-bearing capacity for a moving load, such as an axle. In this study, the axle load is applied directly at the quarter-span of the arch, and the optimization of the arch's dimensions is conducted with respect to this point. The effect of filling material, both in terms of load distribution and pressure, has not been considered to simplify the mechanical model.

5.3. Results and Discussion

The RBDO process was initialized from the geometric dimensions of the existing Dar El Makina bridge, and a reliability-based optimization was conducted. Both solution strategies delivered consistent outcomes for the two design variables and the cost function. In the optimized configuration, the arch thickness is 27% lower than in the original structure, while the rise is 13% higher (Table 5). These values define the final geometry adopted in the remainder of the study.

Table 5. RBDO results

Optimal cost ($\frac{e^*}{2h^*}$)	0.13251
Optimal design ($e^*; h^*$)	(0.4714 m; 1.7787 m)

Using the Reliability Index Approach (RIA) did not lead to conclusive results and the algorithm failed to converge, which we attribute to the computational burden associated with repeated calls to the mechanical model. Similar issues were reported by Chateauneuf & Aoues [24], who compared RIA and PMA on several examples and showed that RIA is less efficient than performance-measure-based formulations. Accordingly, the results discussed in this article correspond to the PMA and SORA approaches.

The PMA and SORA methods exhibit noticeable differences in computational behavior. The SORA strategy converged after four deterministic optimization cycles (Figure 11, Table 6), whereas PMA required 47 iterations to meet the stopping criteria (Figure 12, Table 7). Despite this faster outer-loop convergence, SORA involved a higher number of mechanical model evaluations (about +35% relative to PMA), due to the reliability checks performed within each cycle.

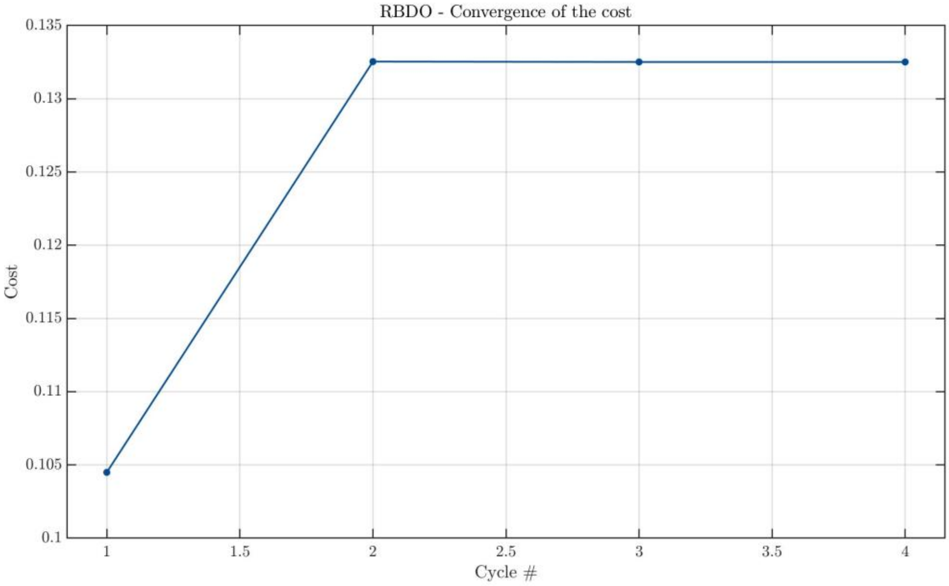

Figure 11. Graphical visualization of the convergence of the SORA approach

Table 6. Results of PMA approach

Number of iterations	47
Number of mechanical model evaluations	77790
Algorithm for inverse FORM	HMV
CPU time	3196

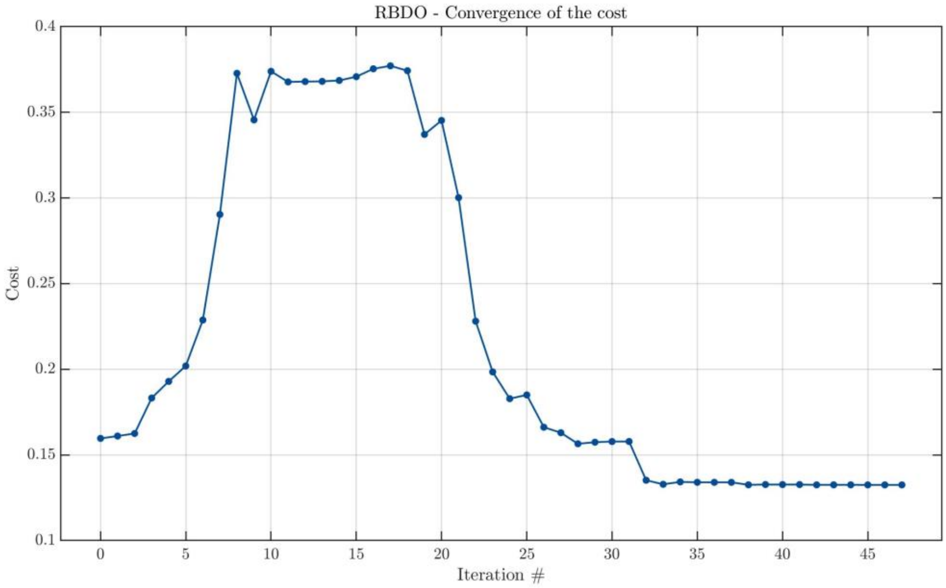

Figure 12. Graphical visualization of the convergence of the PMA approach

Table 7. Results of SORA approach

Number of cycles	4
Number of mechanical model evaluations	104286
Algorithm	SQP
CPU time	1913

Even with the larger number of model evaluations, the total CPU time favoured SORA, which was approximately 40% lower than PMA. This advantage is attributed to the decoupled structure of SORA, which focuses each cycle on a simpler deterministic subproblem guided by the reliability update, instead of executing a fully nested reliability-based optimization at every iteration. This finding is consistent with Lee et al. (2002) [35], who reported faster convergence for SORA compared to two-level PMA formulations.

The Reliability Index Approach (RIA) failed to converge due to the computational demands of the mechanical solver and the nested probabilistic transformations, confirming its limited suitability for nonlinear limit-state functions. To address this issue, Du & Chen (2004) [36] introduced the Sequential Optimization and Reliability Assessment (SORA) method, which decouples reliability assessment from optimization, reducing computational cost while maintaining accuracy. Chateauneuf & Aoues (2008) [24] and López & Beck (2012) [37] later demonstrated that performance-measure-based strategies such as PMA and SORA offer superior numerical stability and efficiency. These observations are consistent with the present results, where both approaches achieved accurate and computationally efficient solutions for the rigid-block analysis of masonry arches.

The final design (thinner section with a higher rise) follows well-established mechanical principles for masonry arches: increasing the rise reduces the horizontal thrust and the tendency to form hinge mechanisms, while decreasing the thickness limits the self-weight without compromising the compressive stress field. These tendencies are consistent with the classic lower-bound and thrust-line formulations developed by Heyman (1966) [26] and extended by Gilbert (2007) [7], which state that an efficient configuration is achieved when the thrust line remains confined within the masonry depth.

To further validate the optimized geometry, the results were compared with the analytical predictions from Nodargi & Bisegna (2020) [8], who revisited the classical thrust-line analysis and proposed a theoretical optimization of the catenary mid-curve under self-weight. As shown in Table 8**,** their findings indicate that a nearly shear-free and bending-moment-free stress state occurs for a rise-to-half-span ratio $h/(l/2) = 0.75$ and a thickness-to-half-span ratio $e/(l/2) = 0.20$. The present reliability-based optimization (RBDO + Rigid Block Analysis) yielded similar proportions, confirming the mechanical consistency of the numerical framework.

Table 8. Comparison of geometric ratios (RBDO + RBA) with Nodargi & Bisegna (2020) [8]

Ratio	Original (Empirical)	Thrust-line analysis [8]	Present RBDO + RBA
$h/(l/2)$	0.675	0.75	0.762
$e/(l/2)$	0.257	0.2	0.202

6. Conclusions

The methodology developed in this paper presents a Reliability-Based Design Optimization (RBDO) framework coupled with a rigid-block mechanical model for the safe and optimal design of a new masonry arch bridge. The original bridge being replaced was constructed following empirical rules, which, while historically effective, did not explicitly account for the stochastic variability inherent in materials, geometry, and loads. The proposed RBDO formulation addresses this limitation by introducing a rational framework in which reliability constraints and optimization are explicitly integrated into the design process.

Among the three RBDO strategies evaluated, the Reliability Index Approach (RIA) failed to converge due to the high nonlinearity of the limit-state function and the computational complexity of the nested probabilistic transformations. This limitation was also observed in benchmark studies where RIA exhibited instability for non-smooth mechanical responses. In contrast, both the Performance Measure Approach (PMA) and the Sequential Optimization with Reliability Assessment (SORA) delivered consistent and physically meaningful results. The comparison between the two approaches confirms that SORA achieves a better compromise between accuracy and computational efficiency, primarily due to its decoupled formulation that separates the deterministic optimization and the reliability assessment. Although SORA requires more mechanical evaluations per cycle, it reduces the total CPU time by approximately 40%, a significant advantage when the mechanical solver is demanding, as in rigid-block formulations.

Although the present study did not employ surrogate modeling, the computational efficiency achieved by combining the rigid-block analysis with PMA and SORA demonstrates the adequacy of direct model evaluations for single-span arches. In future work, adaptive surrogate models such as Kriging or Polynomial Chaos Expansion could be integrated to accelerate reliability-based optimization when dealing with multi-span or materially nonlinear configurations.

The mechanical model implemented in this study is based on the rigid-block limit analysis, a classical and computationally efficient method for assessing the load-bearing capacity of masonry arches. For simplicity, the masonry blocks were assumed to possess infinite compressive strength, thereby neglecting potential crushing phenomena near hinge formations. This assumption, though simplifying, remains acceptable for short to moderate spans and semi-circular geometries, where compressive stresses remain well within safe limits. However, for flatter arches or materials with lower compressive resistance, this simplification may lead to unconservative estimations, suggesting that future developments should incorporate nonlinear compressive constraints to improve accuracy. Similarly, the effect of the backfill was not explicitly modeled in this study. The backfill typically contributes to the mechanical behavior through load dispersion, arching effects, and passive lateral pressures, which enhance the overall stability of the arch. While including these interactions could provide a more realistic estimate of the load-bearing capacity and potentially alter the optimal design, doing so would significantly increase the computational complexity of the coupled RBDO process, particularly when using decoupled strategies like SORA. The inclusion of soil–structure interaction thus remains a promising topic for future research.

In practical applications, the position of the moving axle load plays a critical role in determining the governing limit state. Within the planar model adopted here, the load could theoretically occupy an infinite number of positions along the span, resulting in an intractable number of analysis cases. To address this, only the most critical load positions—those generating the largest bending or thrust effects—were retained for evaluation within each RBDO iteration. This approach ensures computational efficiency while maintaining mechanical relevance. Ideally, a deterministic load-bearing verification should be performed after each optimization cycle to ensure that the updated design parameters satisfy the safety requirements for all relevant load positions within the target reliability level.

Finally, the present framework focused on geometric and loading uncertainties, which are the most influential for the design of new masonry bridges, where material properties can be adequately controlled through construction specifications and quality management. For existing bridges, however, material variability and degradation become predominant sources of uncertainty. This complementary problem is foreseen as a future development of this research, where the proposed framework can be directly extended to the assessment and strengthening issues. For degraded structures, the random variables may include parameters describing deterioration, such as reduced friction coefficients, altered joint stiffness, or geometric imperfections due to settlement. For retrofitted bridges, additional design variables such as injected mortar stiffness or reinforcement parameters can be introduced to model different intervention strategies. In summary, the proposed RBDO approach provides a mechanically consistent, reliability-controlled, and computationally tractable framework for the design of new masonry arch bridges. It bridges the gap between empirical design practices and modern probabilistic standards, offering a scientifically grounded pathway toward the safe, efficient, and sustainable revival of masonry bridge construction.

7. Nomenclatures

AMV	Advanced Mean Value	CDF	Cumulative Distribution Function
CoV	Coefficient of Variation	FORM	First Order Reliability Method
HMV	Hybrid Mean Value	HL-RF	Hasofer and Lind-Rackwitz and Fiessler algorithm
MPP	Most Probable Point	MPTP	Minimum Performance Target Point
PDF	Probability Density Function	PMA	Performance Measure Approach
RBA	Rigid Block Analysis	RBDO	Reliability-Based Design Optimization
SORA	Sequential Optimization with Reliability Assessment	SORM	Second Order Reliability Method
SQP	Sequential Quadratic Programming	$c(\boldsymbol{d})$	Cost function
$\mathbb{D}$	Design space	$\boldsymbol{d}$	Design variables
$\boldsymbol{E}$	Equilibrium matrix	e_0	Original arch thickness
e^*	Optimal arch thickness	$\boldsymbol{F}_D$	Dead loads due to self-weight
$\overline{\boldsymbol{F}_L}$	Target live load	$f(\boldsymbol{d})$	Soft constraint
$f_{\boldsymbol{X}}$	Probability density function of the vector $\boldsymbol{X}$	$\mathrm{G}(\mathbf{u})$	Performance/limit state function in standard space
$g(\boldsymbol{X})$	Performance/limit state function in physical space	h_0	Original arch rise
h^*	Optimal arch rise	l	Span of the bridge
M_d	Dimension of the design space	m_i	Moment at joint i
N	Number of arch blocks	n_i	Normal force at joint i
P_f	Probability of Failure	$\boldsymbol{q}_x$	Contact vector at joint
R	Radius at extrados	s_i	Shear force at joint i
$\mathcal{T}$	Iso-probabilistic transformation	$\tan(\phi)$	Friction coefficient at joint
w	Width of the bridge	$\boldsymbol{X}$	Vector of random variables
$\boldsymbol{Z}$	Environmental variables	Φ	Gaussian cumulative density function
β	Reliability Index	γ_m	Density of masonry blocks
$\zeta_{\overline{\boldsymbol{F}_L}}$	Position of the axle of target live load w.r.t. 1st arch axis	μ	Mean value
Φ	Cumulative gaussian probability distribution function	Ω	Domain of realizations of random variable

8. Declarations

8.1. Author Contributions

Conceptualization, M.T.F.; methodology, M.T.F.; validation, M.R.; writing—original draft preparation, M.T.F.; writing—review and editing, M.R.; supervision, M.E.; project administration R.S. All authors have read and agreed to the published version of the manuscript.

8.2. Data Availability Statement

The data presented in this study are available on request from the corresponding author.

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8.4. Conflicts of Interest

The authors declare no conflict of interest.

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