Comparison between Analytical Equation and Numerical Methods for Determining Shear Stress in a Cantilever Beam

Imad Al-Qasem [a], A. Rasem Hasan [b], Mohanad Y. Abdulwahid [c*], Isaac Galobardes [d]

[a] *Lecturer, Department of Civil Engineering, An-Najah National University, Nablus, Palestine.*

[b] *Assistant Professor, Department of Civil Engineering, An-Najah National University, Nablus, Palestine.*

[c] *Lecturer, Department of Civil Engineering, Faculty of Engineering, Koya University, Danielle Mitterrand Boulevard, Koya KOY45, Kurdistan Region - F.R. Iraq.*

[d] *Lecturer, Department of Civil Engineering, Xi'an Jiaotong-Liverpool University, 111 Ren'ai Road, Suzhou Dushu Lake Higher Education Town, 215123 Suzhou, Jiangsu Province, China.*

Abstract

A three meter-length cantilever beam loaded with a concentrated load at its free end is studied to determine shear stresses. In the present study, three cross sections are considered: rectangle (R); I, and T. The study presents a comparison of maximum shear stresses obtained by means of two methods: classical analytical equation derived by Collingnon, and finite element method (FEM) software. Software programs ANSYS and SAP2000 were used. The results show difference between the maximum shear stresses obtained by the analytical equation and the software, being the last is always higher. The average differences for ANSYS and SAP2000, independently of the cross section, were 12.76% and 11.96%, respectively. Considering these differences, correction factors were proposed to the classical analytical formula for each cross section case to obtain more realistic results. After the correction, the average differences decrease to 1.48% and 4.86%, regardless of the cross section shape.

Keywords: Shear Stress; Finite Element Methods; Analytical Equations; Comparison Analysis; Correction Factor.

1. Introduction

As basic structural elements, beams have been extensively used in different fields such as architecture, mechanics, chemistry, aerospace and ocean engineering [1]. A beam is a structural element that primarily resists loads applied laterally to its axis. In a beam, the internal force system consists of a shear force and bending moment acting on its cross section [2]. The basic courses in materials and structural mechanics focus on determining stresses in beam structures.

The study of beams is not trivial due to the fact that the forces and moments may vary along the length of loaded beams. The internal forces in the element entail two types of stresses considering their transverse section: normal stresses, resulted from applying axial forces and bending moments, and shear stresses, resulted from applying shear forces and torsion moments [2]. Regarding the latter, shear stresses (τ) are commonly determined using the classical analytical equation defined by Collignon [3] (Equation 1). This equation is used for straight prismatic members made of homogeneous material that presents a linear elastic behavior and whose internal resultant shear force is directed along an axis of symmetry of the cross-sectional area [2]. Collignon's formula measures the shear stress considering

the shear force (V); the width and the moment of inertia of the cross section of the element (t and I, respectively), and the first moment of area in respect of the neutral axis (Q).

$$\tau = \frac{V \cdot Q}{I \cdot t} \tag{1}$$

Collingnon derived his shear stress formula at the end of the 19th century. Later, the evolution of computing sciences, and the employment of the finite element methods (FEM) in structural analyses, urge the review of this formula. The FEM is a numerical method that can be used for the accurate solution of complex engineering problems [4]. Models based on FEM, easily allow identifying the combination of material properties or dimensions of elements that maximize the performance of a structure considering certain requirements. Furthermore, modern calculation models based on FEM enable the consideration of all the unavoidable initial imperfections that a design might present and are not considered when using analytical equations [5].

Both methodologies, analytical equations and FEM, are used to estimate the values of stresses in a structure. Results may vary across the used method. This variation should be identified, since during a design of a same structure both methodologies could be used simultaneously. Recent studies focused on comparing the differences obtained by measuring the shear stresses with FEM and analytical equations. For instance, researchers studied FEM for shear stress for a cantilever beam subjected to a concentrated load at its free end [6]. The stresses were calculated by using a FEM software, ANSYS (American computer-aid engineering software) for different elements: beams, shells, planes and solids. Results showed differences between the analytical and modeled solutions for each element. These differences got values up to 158.27%, presented by the element solid; being the values obtained using the FEM always higher than the ones obtained with the analytical equation. Apart from that, the maximum shear stresses in a beam also using ANSYS and Collingnon equation were studied [7]. Results obtained by ANSYS showed differences up to 10%, and higher than the analytical method as well. Hence, the findings of these researchers showed that considering the stresses obtained with the FEM the engineers would produce a safer design.

Although there are a huge number of FEM software programs, differences in calculation between them are expected. For instance, SAP2000 is a FEM engineering simulation software highly used for civil engineering designs and there is not literature regarding the presented issue.

2. Methodology

In this study, three dimensional linear finite element models of concrete beams have been built by using both ANSYS 8.0 and SAP2000. The particular case of prismatic concrete cantilever beam subjected to a punctual load on the free end was investigated to determine the maximum shear stress. Three different cross sections (rectangular, I, and T) were considered to study how the geometry affects the results.

The numerical results of FEM were compared to those obtained by the analytical equation (Equation 1) and percentage differences were also calculated. Finally, based on the differences between the analytical method and FEM models, correction factors were proposed to the classical equation and the goodness of fit of the corrected equation was determined.

3. Case Study

3.1. Definition of the Structural Element

To simplify the case study, a three meter length cantilever subjected to a punctual load (P) at the free end is considered (Figure 1a). The self-weight of the beam is ignored in order to study the effect of shear force on cross section without any side effect come from other forces. Notice that simplifying the case, torsion moments and axial forces are neglected and the diagram of the shear forces is constant along the length of the cantilever (Figure 1b).

Three different cross section areas, which are the most used concrete cross sections [8], were analyzed: rectangular (R), I, and T. The geometry of each section is presented in Figure 1c.

3.2. Maximum Shear Stresses Assessment

3.2.1. Analytical Equation

The maximum shear forces were calculated by using Collignon formula for each cross section area case. Equation 1 is calculated in order to have only one unknown; the punctual load P. As an example, the R cross section case is presented.

Considering the shear force diagram presented in Figure 1b, the shear force (V) is equal to the punctual load P. Given the geometry of the rectangle, the width (t) was taken as 100 mm. A^* is the area of the top (or bottom) portion of

the members cross-section area, above (or below) the section plane where is (t) is measured [2]. The moment of inertia of the cross section (I) and the first moment of area (Q) in respect of the neutral axis (y'), which is the distance from the neutral axis to the centroid of (A) [2], were calculated by using Equation 2 and 3, respectively.

$$I = \frac{bh^3}{12} \tag{2}$$

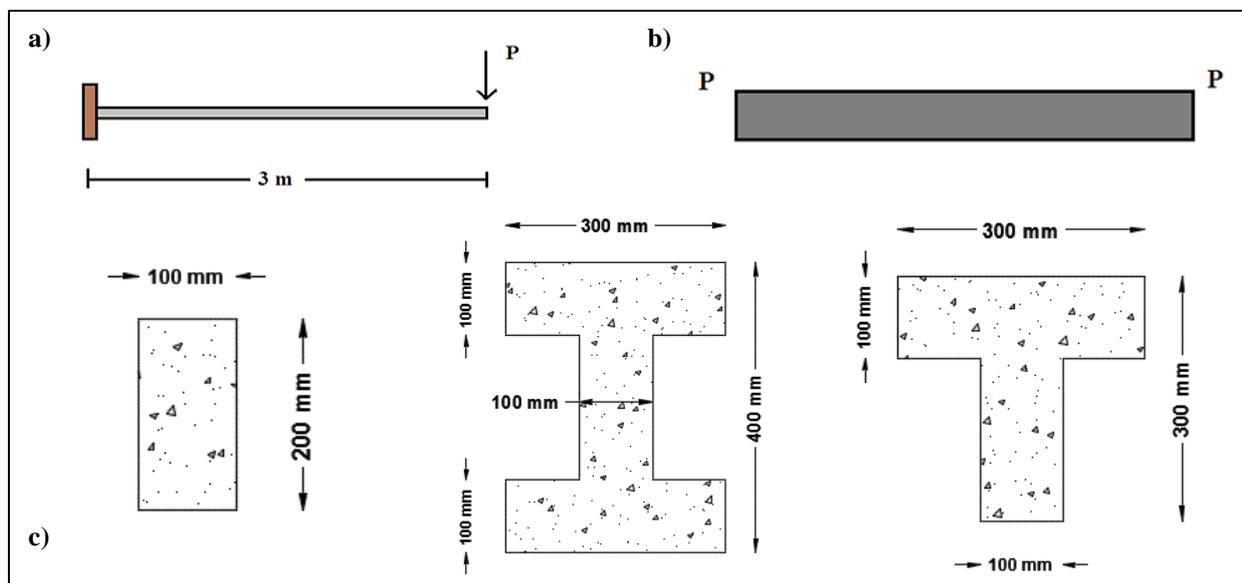$$Q = A^*.y' = b.\frac{h}{2}.y' \tag{3}$$

Figure 1. (a) Cantilever; (b) shear force diagram; (C) Cross-section areas

Then, Equation 4 defines the maximum shear stress (τ_{max}) for the cantilever with R cross section based on the punctual load P. Notice that P is the variable considered in this study. Equation 5 and Equation 6 measure τ_{max} for the cross section I and T cases, respectively. Units of P and τ_{max} are kN and MPa, respectively.

$$\tau_{max} = 7.5 \cdot 10^{-2} \cdot \text{P} \tag{4}$$

$$\tau_{max} = 3.4 \cdot 10^{-2} \cdot \text{P} \tag{5}$$

$$\tau_{max} = 4.9 \cdot 10^{-2} \cdot \text{P} \tag{6}$$

3.2.2. Finite Element Modeling and Material properties

ANSYS and SAP2000 were used as FEM software to measure the maximum shear stresses in the studied cantilever beams. In order to represent a structural model for the cantilever beam, the following steps were applied independently on the used software:

- Three cantilever beams, one for each section considered, were built using a 3D modeler. These modelers are more natural since, and represent real structures with 3D solid elements [9].

- Constraints were applied to the fix support of the cantilever in order to avoid translations and rotations.

- Concrete with a characteristic compressive strength (f_c') of 28 MPa and a modulus of elasticity (E_c) equal to 29 GPa were applied to the all beams models. This material was chosen because the majority of normal concretes present a characteristic compressive strength from 15 to 45 MPa [10]. Notice that E_c was calculated by using the equation presented in Eurocode 2 [11].

- Meshing was applied to the element. Notice that the mesh was optimized for both the before mentioned FEM software. Consequently, a smaller size of mesh would not give a more adjusted result (this result should me mentioned in the results). Table 1 presents the general meshing features used in the study, whereas Figure 2 shows the beams created with SAP2000 as an example.

- The load P was applied to the model. P is the studied parameter in this study as mentioned before.

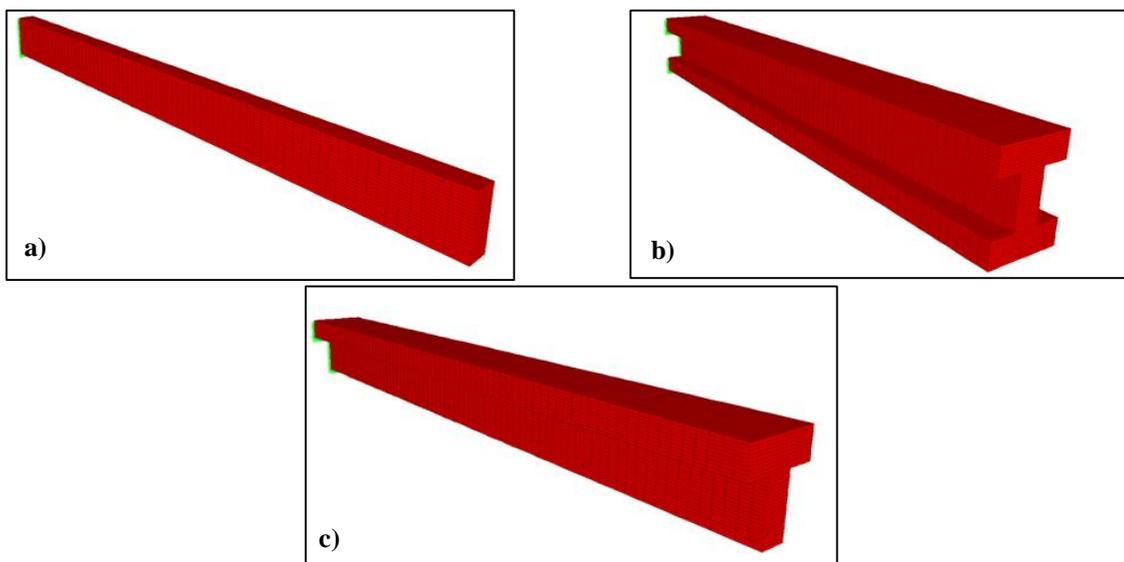- The model was run and the results obtained.

Figure 2. Meshing of the cantilever: (a) R; (b) I; (c) T crosses sections

Table1. Description of Meshing of the Cantilever

Section	Information
R	Total number of nodes: 17556 Total number of elements: 15000 The Size of each element: 40×10×10 mm
I	Total number of nodes: 67716 Total number of elements: 60000 The Size of each element: 40×10×10 mm
T	Total number of nodes: 42636 Total number of elements: 37500 The Size of each element: 40×10×10 mm

4. Results

4.1. Analysis Results by Analytical Equation

Table 2 presents the results of maximum shear stresses (τ_{max}) measured by using the analytical equations derived for each studied cross sections. These cross sections were estimated by using Equation 4, 5 and 6 for R, I, and T cross sections, respectively. τ_{max} and load values (P) equal to 50, 100, 200, 300, 400 and 500 kN.

Table 2. Maximum shear stress (τ_{max}) assessed using the analytical equation (Unit: MPa)

Section	P (kN)					
	50	**100**	**200**	**300**	**400**	**500**
R	3.75	7.50	15.00	22.50	30.00	37.50
I	1.70	3.40	6.80	10.20	13.60	17.00
T	2.50	4.99	9.98	14.97	19.96	24.95

4.2. Analyses Results by Numerical Methods

Figure 3 presents, as an illustrative example, the obtained maximum shear stress results by using SAP2000 for each studied cross section. Notice that all beams are predominantly shown in one unique color. This phenomenon was expected since the shear forces and the cross section of the beams are constant along its length. However, two regions present slightly different colors: the fix end of the beam and the other extreme where the load is applied. This change is due to the Saint Venant's principle that entails a stress distortion on the boundary conditions [12].

Table 3 gathers the maximum shear stresses (τ_{max}) obtained with the ANSYS and SAP2000. The results are organized by cross section and value of load applied (P). P adopted the same values as for the analytical equation estimations.

On one hand, the results obtained with ANSYS and SAP2000 present differences. In average, the τ_{max}, by using ANSYS achieved 5.63% and 4.88% higher than the one obtained by using SAP2000 for cross sections R and T, respectively. On the other hand, the results of modeled section I were 8.81% higher than those of SAP2000. However, these difference among obtained results are not statistically significant based on a t-test (p-value of 0.092 > 0.05).
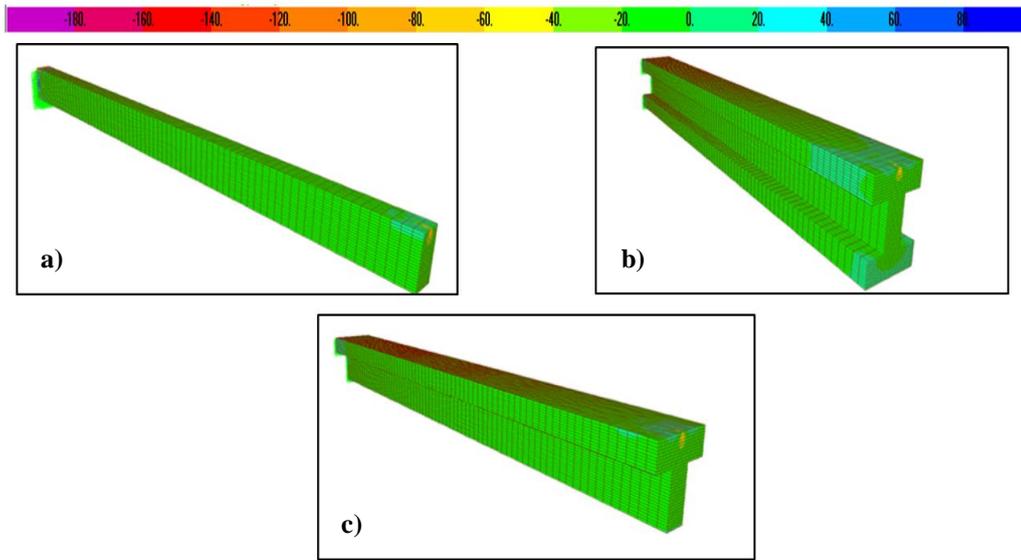

Figure 3. Shear stresses distribution: (a) R (b) I (c) T crosses sections

Table 3. Maximum shear stress (τ_{max}) assessed using FEM (Unit: MPa)

Software	Section	P (kN)					
		50	100	200	300	400	500
ANSYS	R	4.39	8.79	17.59	26.39	35.19	43.99
	I	1.91	3.82	7.64	11.57	15.29	19.11
	T	2.85	5.69	11.39	17.08	22.78	28.47
SAP2000	R	4.14	8.29	16.59	24.89	33.18	41.48
	I	2.07	4.15	8.30	12.45	16.60	20.75
	T	2.71	5.42	10.84	16.26	21.68	27.11

4.3. Comparison between the Analytical Equation and Numerical Methods

Table 4 presents the differences in percentage between the τ_{max} results obtained with the analytical equation and the ones achieved by using the FEM models. Notice that the results obtained with the FEM are higher than the ones achieved with the analytical equation, independently of the software used and the cross section of the beam, similar behavior to former researches [13, 14]. Using the software ANSYS, the average differences were 14.70%, 11.15% and 12.40% for the cross sections R, I and T, respectively. For SAP2000, these differences were 9.64%, 18.37% and 7.86%, respectively. Even though the highest difference is obtained by using SAP2000 for the cross section I, this software presents, independently of the cross section, lower average difference (11.96%) than the one presented by ANSYS (12.76%).

Table 4. Differences of maximum stress (τ_{max}) between analytical and FEM results (Unit: %)

Software	Section	P (kN)					
		50	100	200	300	400	500
ANSYS	R	17.07	17.2	17.27	17.29	17.3	17.31
	I	12.35	12.35	12.35	13.43	12.43	12.41
	T	14	14.03	14.13	14.09	14.13	14.11
SAP2000	R	10.4	10.53	10.6	10.62	10.6	10.61
	I	21.76	22.06	22.06	22.06	22.06	22.06
	T	8.40	8.62	8.62	8.62	8.62	8.66

The relationship between the results obtained with the analytical equation (τ_{max}, AE) and the ones obtain by using the FEM software (τ_{max}, FEM) are presented in Figure 4. The graphed pairs of values show a coefficient of variation (R^2) of 0.999 and 0.997 for ANSYS and SAP2000, respectively. However, the differences between them are statistically significant, showing p-values lower than 0.01 ($4 \cdot 10^{-5}$ and $4 \cdot 10^{-6}$, respectively). Therefore, this result highlights the necessity to adjust the analytical equation based on the results obtained with the FEM software.
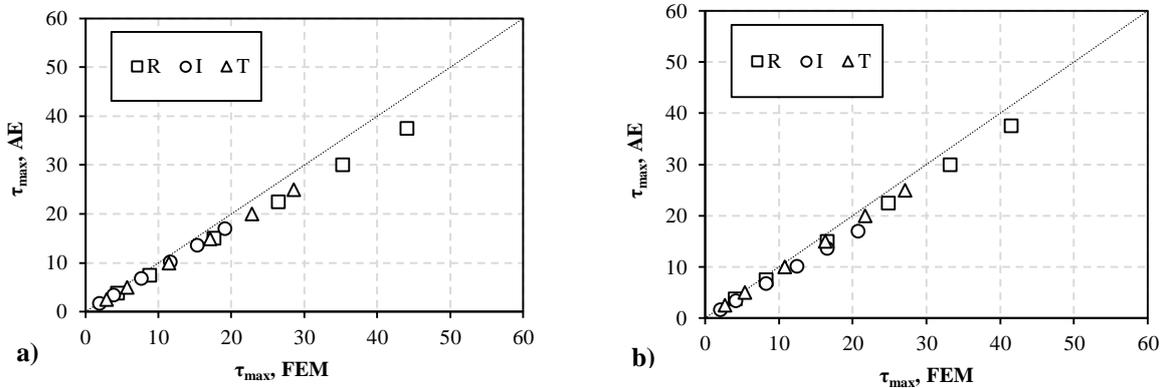

Figure 4. Comparison between τ_{max}, AE and τ_{max}, FEM: (a) ANSYS and (b) SAP2000

4.4. Correction Factors

As mentioned before, the analytical equation should be adjusted considering the FEM software results. Thus, a correction factor α is introduced to the analytical equation (Equation 7). Values of α were determined by Using a free-access experimental data curve fitting software (LAB Fit), methodology is described in former researches [13, 14, 15]. Table 5 presents the values of the correction factors α considering different analyzed cases: (i) all results together independently of the FEM software used and the cross section of the beams (Global); (ii) all results obtained separately by ANSYS and SAP2000 independently of the cross sections (All ANSYS and All SAP2000, respectively; and (iii) the cases considering both FEM software and cross section. The coefficient of variation (R^2) of the results using the correction factor is also presented for each case.

$$\tau_{max} = \alpha . \left(\frac{V.Q}{I.t} \right) \tag{7}$$

Table 5. Correction factors for the empirical approach

| Case		α	R^2
Global		1.137	0.997
	All	1.159	0.999
	R	1.173	1.000
ANSYS	I	1.126	0.999
	T	1.141	0.999
	All	1.115	0.996
	R	1.106	1.000
SAP2000	I	1.221	0.999
	T	1.086	0.999

Regarding the correction factors, the maximum and minimum values are 1.221 and 1.086, respectively. These are obtained for the results achieved with SAP2000 and cross sections I and T, respectively. Regarding the used software, ANSYS and SAP2000 presented values of α equal to 1.173 and 1.115, respectively, independently of the cross section of the analyzed beam. Finally, the α obtained using all the results (Global case) is 1.137. The authors consider the last value as the correction factor since it considers different FEM software and cross sections.

Table 6 presents the new differences in percentage between the τ max results obtained with the corrected analytical equation and the ones achieved using the FEM software. Notice that using the corrector factor, the table shows negative values. This means that the corrected analytical results are higher than the ones obtained with the FEM software. Using the software ANSYS, the average differences were 3.11%, -1.01% and 0.34 % for the cross sections R, I and T, respectively. While for SAP2000, these differences were -2.76%, 7.31% and -4.50%, respectively. In absolute average, ANSYS and SAP2000 present difference of 1.48% and 4.86%, respectively.

The new relationship between the results obtained with the corrected analytical equation (τ_{max}, CAE) and the ones obtained using the FEM software (τ_{max}, FEM) are presented in Figure 5. The graphed pairs of values show a coefficient of variation (R^2) of 0.999 and 0.998 for ANSYS and SAP2000, respectively. After the correction, the differences between them are not statistically significant. Therefore, this highlights the goodness of the correction.

Table 6. Differences of maximum stress (τ_{max}) between corrected analytical and FEM results (Unit: %)

Software	Section	P (kN)					
		50	100	200	300	400	500
ANSYS	R	2.96	3.08	3.14	3.16	3.17	3.17
	I	-1.18	-1.18	-1.18	-0.24	-1.12	-1.13
	T	0.26	0.29	0.38	0.35	0.38	0.36
SAP2000	R	-2.90	-2.79	-2.73	-2.71	-2.73	-2.71
	I	7.09	7.35	7.35	7.35	7.35	7.35
	T	-4.66	-4.47	-4.47	-4.47	-4.47	-4.44
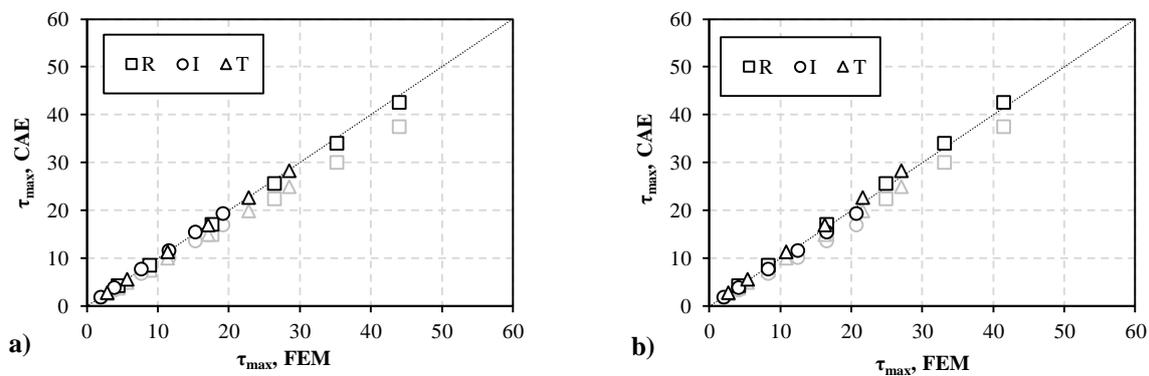
Figure 5. Comparison between τ_{max}, CAE and τ_{max},FEM: (a)ANSYS and (b)SAP2000

5. Conclusion

The main conclusion of this study is that the use of finite element models is more accurate than the use of traditional equations to measure the shear stress in a beam cross section. In this research, a difference of 14% was found, and this calls attention to the need of correction. Considering the type of finite element model and the cross section of the beam, the differences in shear stress values obtained by using ANSYS were 17%, 14%, and 12%, and using SAP2000 these were 11%, 9%, and 22%, for R, T, and I sections, respectively. Results of maximum shear stress obtained for all cross sections by ANSYS and SAP2000 were significantly different and greater than the values obtained from the Collingnon's equation. . Therefore, cross sections will be exposed to higher shear stresses than the ones used for the design based on analytical equations, and this may lead to structural failure. To overcome this design deficiency, safety factor for maximum shear stress is proposed based on the results obtained using FEM, and taken as global value of 1.137, regardless of the cross section of cantilever beams (Equation 7).

5. References

[1] Duan T. C., Li L. X. "The Unified Solution for a Beam of Rectangular Cross-Section with Different Higher-Order Shear Deformation Models." Latin American Journal of Solids and Structures 13 (September 2017): 1716-1737. doi:10.1590/1679-78252732.

[2] Ghavami P. "Mechanics of Materials" (2015). doi.org/10.1007/978-3-319-07572-3.

[3] Kurrer, K. E. "The History of the Theory of Structures: From Arch Analysis to Computational Mechanics." (2012). doi.org/10.1111/j.1475-1305.2009.00638.x.

[4] Singiresu S. R. "The Finite Element Method in Engineering.4th" (2005). doi.org/10.1016/b978-0-7506-7828-5.x5000-8.

[5] Martinásek J., Valeša, J.,Kala Z. "Inelastic finite element analysis of lateral buckling for beam structures." Procedia Engineering 172 (2017):481 – 488. doi:10.1016/j.proeng.2017.02.056.

[6] Yeghiazarian, H., B. Yazdizadeh."Some FEM Models for Bending and Vibration Problems of Beam." Proceedings of the Yerevan State University, Physical and Mathematical Sciences. (2013): 38–43.[7] Thompson M., Thompson, J. "ANSYS

Mechanical APDL for Finite Element Analysis." (2017):123–130 doi.org/10.1016/B978-0-12-812981-4.00018-6.

[8] Ruiz M., González C. "Influence of flanges on the shear-carrying capacity of reinforced concrete beams without web reinforcement." Structural concrete 18 (March 2017): 720-732. doi:10.1002/suco.201600172.

[9] P. Kettil, N. E. Wiberg. "Application of 3D solid modelling and simulation programs to a bridge structure." Engineering with Computers 18 (August 2002):160-169. doi:10.1007/s003660200014.

[10] Qasrawi, H. "Design of Normal Concrete Mixtures Using Workability-Dispersion-Cohesion Method." Advances in Civil Engineering (2016): 1-11. doi:10.1155/2016/1035946.

[11] EN 1992-1-1 (English): Eurocode 2: Design of concrete structures - Part 1-1: General rules and rules for buildings [Authority: The European Union Per Regulation 305/2011, Directive 98/34/EC, Directive 2004/18/EC]. (2004). doi: 10.3403/03178016u.

[12] Love, A. E. H."A Treatise on the Mathematical Theory of Elasticity, Edition 4th" (2011). doi.org/10.2307/2300013.

[13] Galobardes I. et al. "Estimation of the modulus of elasticity for sprayed concrete." Construction and Building Materials 53(February 2014):48-58. doi:10.1016/j.conbuildmat.2013.11.046.

[14] Galobardes, I., Figureiredo, A. "Correlation between beam and Barcelona tests for FRC quality control for structural applications." Proceedings of the International Conference FIBRE CONCRETE. (2015):149-158. doi 10.13140/RG.2.1.4771.6327.

[15] Sandoval, G. F. B. et al. "Comparison between the falling head and the constant head permeability tests to assess the permeability coefficient of sustainable previous concrete." Case Studies in Construction Materials 7 (December 2017): 317-328. doi:10.1016/j.cscm.2017.09.001.