Available online at www.CivileJournal.org

Civil Engineering Journal

Vol. 6, No. 4, April, 2020

Approximation of the Multidimensional Optimal Control Problem for the Heat Equation (Applicable to Computational Fluid Dynamics (CFD))

Yuri Alexandrovich Kostikov [a], Alexander Mikhailovich Romanenkov [b, c*]

[a] *Candidate of Physical and Mathematical Sciences, Head of Department 812, Moscow Aviation Institute (National Research University), 4, Volokolamskoe Shosse, GSP-3, A-80, 125993, Moscow, Russia.*

[b] *Candidate of Technical Sciences, Associate Professor, Department of 812, Moscow Aviation Institute (National Research University), 4, Volokolamskoe Shosse, GSP-3, A-80, 125993, Moscow, Russia;*

[c] *Senior Researcher, Federal Research Center "Informatics and Management" of the Russian Academy of Sciences, 9, prospect 60-letiya Oktyabrya, Moscow, Russia.*

Received 23 January 2020; Accepted 14 March 2020

Abstract

This work is devoted to finding an estimate of the convergence rate of an algorithm for numerically solving the optimal control problem for the three-dimensional heat equation. An important aspect of the work is not only the establishment of convergence of solutions of a sequence of discrete problems to the solution of the original differential problem, but the determination of the order of convergence, which plays a very important role in applications. The paper uses the discretization method of the differential problem and the method of integral estimates. The reduction of a differential multidimensional mixed problem to a difference one is based on the approximation of the desired solution and its derivatives by difference expressions, for which the error of such an approximation is known. The idea of using integral estimates is typical for such problems, but in the multidimensional case significant technical difficulties arise. To estimate errors, we used multidimensional analogues of the integration formula by parts, Friedrichs and Poincare inequalities. The technique used in this work can be applied under some additional assumptions, and for nonlinear multidimensional mixed problems of parabolic type. To find a numerical solution, the variable direction method is used for the difference problem of a parabolic type equation. The resulting algorithm is implemented using program code written in the Python 3.7 programming language.

Keywords: Approximation of a Three-Dimensional Parabolic Problem; Optimal Control; Convergence of the Gradient Method; Integral Estimates; Functional Convergence Estimation, CFD.

1. Introduction

The heat equation is used to find the dependence of the temperature of the medium on the spatial coordinates and time, for given coefficients of heat capacity and heat conductivity. This is a second order partial differential equation, which is a parabolic type equation. Since the need to determine temperatures in the whole space is often absent, when setting the problem, additional conditions are introduced that determine the restrictions on the solution of the problem for a given area. For example, one of these conditions is to set the temperature distribution at the boundary of the region (the Dirichlet problem).

* Corresponding author: romanaleks@gmail.com

http://dx.doi.org/10.28991/cej-2020-03091506

The process of finding the temperature distribution at given times is a laborious task. Since differential problems of a continuous nature cannot be programmed due to the limited capabilities of computer technology, such problems, by discretizing them, reduce to similar difference problems. Such a transition is carried out using difference schemes.

The main task of approximation is to find such an approximate function that least, in a certain sense, deviates from a given continuous function. Due to the fact that when solving continuous problems, the differential operators are replaced by finite-difference analogues, which are written in the form of algebraic equations, problems arise for determining the convergence and approximation error.

Note that when switching from a differential operator to a finite-difference analogue, a numerical solution is obtained that differs from the original solution. In such cases, an analysis is performed that determines the approximation order. For example, in Godunov and Ryabenkii (1987) study, the one-dimensional optimal control problem of the heat conduction process and the gradient descent method are considered, on the basis of which the approximation order of the finite-difference problem was obtained [1]. The optimality criterion is based on the gradient descent method, ideas leading to the assertion of the type of maximum principle by L. S. Pontryagin [2-4], lead to significant complications and are not considered in this paper. Approximation of optimization problems is considered by many researchers. An important work is Serovaiskii (2013) [5], from which methods for obtaining estimates of the boundedness of the target functional are used.

In modern works, attention is paid to the convergence of functionals in optimization problems of different nature. A hyperbolic boundary value problem with a quadratic cost functional is considered in Edalatzadeh et al. (2020) study [6]. An important point is the use of a similar technique of integral estimates to obtain optimal control in an explicit form. Criteria for the existence of optimal forms in Banach spaces were established in Edalatzadeh (2016 and 2019) studies [7, 8]. For a differential operator in divergent form and for an integro-differential operator in Deligiannidis et al. (2020) and Mukam and Tambue (2020) researches [9, 10] using integral estimates in suitable spaces, weak convergence of the numerical method was established and the order of convergence of the functional sequence to the solution was found.

The technique developed in this paper will be transferred to parabolic problems with variable coefficients, as well as to nonlinear cases. The possibility of such a step is considered plausible due to the Guillén-González et al. (2020) and Biccari et al. (2020) works [11, 12].

The aim of this work is to estimate the approximation of a finite-difference analogue for the heat equation of three spatial variables. The solution to the difference problem is constructed using the variable direction method.

Note that the original result on the convergence estimation of the sequence of the target functional in 3-dimensional space is established and the constants in the O symbols are directly calculated. We can briefly formulate the sequence of actions and steps that are used in the work:

- Statement of the differential problem;
- Analysis of the differential problem, obtaining an estimate of the norm of the solution depending on the control function;
- Building a sequence of discrete tasks;
- Obtaining expressions for errors between solutions to differential and discrete problems;
- Estimation of errors using the technique of Sobolev spaces and the establishment of target inequality.

Based on the discretization of the three-dimensional heat conduction problem, a numerical algorithm is developed, with the help of which a software package is created to determine the time required for uniform distribution of heat in the rod.

2. The Problem Statement

The following is a third-order differential heat equation that describes the process of heating a body in space:

$$\frac{\partial f}{\partial t} = a^2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + u(x,y,z,t)$$
$$(x,y,z,t) \in Q = Q_3 \times (0,T), Q_3 = (0,l_x) \times (0,l_y) \times (0,l_z) \tag{1}$$

For which the following boundary conditions are given;

$$\left.\frac{\partial f}{\partial Q_3}\right|_{\partial Q} = \psi(x,y,z,t), 0 < t < T;$$
$$f|_{t=0} = \varphi(x,y,z), 0 \le x \le l_x, 0 \le y \le l_y, 0 \le z \le l_z, \tag{2}$$

Where $f(x, y, z, t)$- is the solution of the boundary value problem, $u(x, y, z, t) -$ is a control function that shows the temperature at the point (x, y, z), at the moment of time t, $a^2 -$ is the thermal conductivity coefficient, $\varphi(x, y, z) -$ is the temperature of the rod at the initial moment of time at each point, $\psi(x, y, z, t) -$ is a given function from $L_2[(0, l_x) \times (0, l_y) \times (0, l_z)]$. Questions of representation of solutions, existence and uniqueness are stated in Vladimirov (1981) and Shubin (2003) works [13, 14].

We denote that the control belongs to the following set:

$$U = \left\{ u(x, y, z, t) \,\epsilon\, L_2(Q): \int_Q u^2(x, y, z, t) dxdydzdt \leq R^2 \right\}, \tag{3}$$

Where $R = const > 0$.

Such a problem is called the Dirichlet problem or the first boundary value problem. We find a numerical solution to this problem using numerical methods, namely, the finite difference method. By expanding the function in a Taylor series, the first and second partial derivatives are expressed, and the boundary conditions are used to determine the value of the nodes on the boundary region.

The task is to find a function $f(x, y, z, t; u)$, such that on the whole region $L_2[(0, l_x) \times (0, l_y) \times (0, l_z)]$ by the time T we get the distribution function heat close to the given function $b(x, y, z)$. The criterion for this difference problem has the form:

$$J(u) = \int_0^{l_x} \int_0^{l_y} \int_0^{l_z} |f(x, y, z, T; u) - b(x, y, z)|^2 dxdydz \to inf, u \,\epsilon\, U \tag{4}$$

And the boundary conditions are rewritten as follows:

$$\begin{aligned} &\frac{\partial f}{\partial Q_3} = 0, 0 < t < T \\ &f|_{t=0} = 0, 0 \leq x \leq l_x, 0 \leq y \leq l_y, 0 \leq z \leq l_z \end{aligned} \tag{5}$$

In this case, it is necessary to go to the finite-difference analogue of the function $f(x, y, z, T; u)$ and evaluate the approximation order.

3. Equation of a Parabolic Type

In this paper, we consider the process of temperature distribution over a three-dimensional rod with a length, height, and width equal to l_x, l_y, l_z, respectively, for the time interval T, which is described by the heat equation. An inhomogeneous equation is considered:

$$\frac{\partial f}{\partial t} = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) + u(x, y, z, t), \tag{6}$$

Which has coefficient $a^2 = 1$ and boundary conditions (5).

We will seek a generalized solution to the original problem in the form of an expansion into a triple Fourier series. Let:

$$f(x, y, z, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} X_n(x) * Y_m(y) * Z_k(z) * T_{nmk}(t) \tag{7}$$

$$u(x, y, z, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} X_n(x) * Y_m(y) * Z_k(z) * U_{nmk}(t) \tag{8}$$

Substituting these series in Equation (6), we can conclude that (6) is certainly satisfied if the terms of the series are equal for the corresponding indexes of the number series of the left and right sides of the equation:

$$\begin{aligned} &X_n(x) * Y_m(y) * Z_k(z) * T'_{nmk}(t) = \\ &= \left(X''_n(x) Y_m(y) Z_k(z) + X_n(x) Y''_m(y) Z_k(z) + X_n(x) Y_m(y) Z''_k(z) \right) T_{nmk}(t) \\ &\qquad + X_n(x) Y_m(y) Z_k(z) U_{nmk}(t) \end{aligned} \tag{9}$$

By removing the inhomogeneous additive in Equation (9), divide it into $X_n(x) * Y_m(y) * Z_k(z) * T_{nmk}(t)$ and rewrite it in the following form:

$$\begin{cases} X_n''(x) + \lambda^2 X_n(x) = 0 \\ Y_m''(y) + \mu^2 Y_m(y) = 0 \\ Z_k''(z) + p^2 Z_k(z) = 0 \\ T'(t) + (\lambda^2 + \mu^2 + p^2)T(t) = 0 \end{cases} \tag{10}$$

We find a solution to the three Sturm-Liouville problems. We start with the problem $X_n''(x) + \lambda^2 X_n(x) = 0$, with $X_n(0) = X_n(l_x) = 0$. Consider 3 cases of solving a linear differential equation.

For $\lambda^2 < 0$, the general form of the solution takes the form $X_n(x) = C_1 e^{-\lambda x} + C_2 e^{\lambda x}$. Due to the boundary conditions, the solution becomes trivial. This solution does not fit.

For $\lambda^2 = 0$, the general solution is $X_n(x) = C_1 + C_2 x$. The solution, by analogy with the case $\lambda^2 < 0$ also does not fit.

For $\lambda^2 > 0$, the general solution is $X_n(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$. It follows from the boundary conditions that $C_2 = 0$ and $C_1 \sin(l_x x) = 0$. It follows that $\lambda l_x = \pi n$. Consequently, the general decision takes the following form.

$$X_n(x) = C_1 \cos\left(\frac{\pi n}{l_x} x\right), n = 1,2 \ldots \tag{11}$$

To obtain a complete orthonormal system, we define C_n. To do this, take the scalar product from (9), equate it to 1 and find the integral.

$$\int_0^{l_x} C_1^2 \cos^2 \frac{\pi n}{l_x} dx = 1$$

We get that $C_1 = \sqrt{\frac{2}{l_x}}$ and:

$$X_n(x) = \sqrt{\frac{2}{l_x}} \cos\left(\frac{\pi n}{l_x} x\right), n = 1,2 \ldots \tag{12}$$

Similarly, we find a generalized solution for $Y(y)$ and $Z(z)$.

$$Y_m(y) = \sqrt{\frac{2}{l_y}} \cos\left(\frac{\pi m}{l_y} y\right), \text{at } m = 1,2 \ldots \tag{13}$$

$$Z_k(z) = \sqrt{\frac{2}{l_z}} \cos\left(\frac{\pi k}{l_z} z\right), \text{at } k = 1,2 \ldots \tag{14}$$

We find the general solution of the differential equation based on λ^2, μ^2 and p^2.

$$T_{nmk}'(t) + \left(\left(\frac{\pi n}{l_x}\right)^2 + \left(\frac{\pi m}{l_y}\right)^2 + \left(\frac{\pi k}{l_z}\right)^2\right) T_{nmk}(t) = U_{nmk}(t) \tag{15}$$

We apply the variational constant method. We solve the corresponding homogeneous equation and find a generalized solution in which C_n is an arbitrary constant on t.

$$\begin{aligned} T(t) &= C_n(t) e^{-\delta^2 t} \\ T'(t) &= C_n'(t)\, e^{-\delta^2 t} - \delta^2 C_n(t) e^{-\delta^2 t} \end{aligned}, \text{where } \delta^2 = \left(\frac{\pi n}{l_x}\right)^2 + \left(\frac{\pi m}{l_y}\right)^2 + \left(\frac{\pi k}{l_z}\right)^2 \tag{16}$$

We put this in Equation (9) and we obtain that for the unknown function $C_n(t)$ the equality $C_n'(t) e^{-\delta^2 t} = U_{nmk}(t)$ must be satisfied. We get that.

$$C_n(t) = \int_0^t e^{\delta^2 \tau} U_{nmk}(\tau) d\tau \tag{17}$$

Whence the solution of the Cauchy problem is given by the formula:

$$T_{nmk}(t) = \int_0^t e^{\delta^2 (t-\tau)} U_{nmk}(\tau) d\tau, t > 0$$

We obtain a formula for calculating the expansion coefficients $u(x,y,z,t)$ in eigenfunctions. Given the orthogonality of the Sturm-Liouville problem, we obtain:

$$U_{nmk}(t) = \sqrt{\frac{8}{l_x l_y l_z}} \int_{\overline{Q_3}} U(x,y,z,t) \cos\left(\frac{\pi n x}{l_x}\right) \cos\left(\frac{\pi m y}{l_y}\right) \cos\left(\frac{\pi k z}{l_z}\right) dxdydz \tag{18}$$

Hence, on the basis of (7), (9), (11) and (12), we obtain:

$$f(x,y,z,t) = \frac{8}{l_x l_y l_z} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \cos\left(\frac{\pi n x}{l_x}\right) \cos\left(\frac{\pi m y}{l_y}\right) \cos\left(\frac{\pi k z}{l_z}\right) \cdot$$
$$\cdot \int_0^t e^{\delta^2 (t-\tau)} U_{nmk}(\tau) d\tau, \tag{19}$$

Where $\delta^2 = \left(\frac{\pi n}{l_x}\right)^2 + \left(\frac{\pi m}{l_y}\right)^2 + \left(\frac{\pi k}{l_z}\right)^2$, and $U_{nmk}(t)$ is equal to (14).

4. Discretization of the Problem

The difference minimization problem has the following form. It is necessary to minimize the objective function $f(x,y,z,t)$ on the four-dimensional domain $\bar{Q} = [0, l_x] \times [0, l_y] \times [0, l_z] \times [0, T]$, where x, y, z are spatial variables and t is time variable. The grid $w_{h_x h_y h_z \tau} = \{(x_i, y_j, z_k, \tau_p): x_i = i h_x, y_j = j h_y, z_k = k h_z, t_p = p\tau, i = \overline{0..X_h}, j = \overline{0..Y_h}, k = \overline{0..Z_h}, p = \overline{0..P}\}$, where h_x, h_y, h_z, τ are given grid steps, $h_x X_h = l_x, h_y Y_h = l_y, h_z Z_h = l_z, \tau P = T$. Following works [15] and [16] we perform discretization and obtain difference problems.

We define the function $f_{h_x h_y h_z \tau} = \{f_{ijkp}: i = \overline{0..X_h}, j = \overline{0..Y_h}, k = \overline{0..Z_h}, p = \overline{0..P}\}$ on the grid partition $w_{h_x h_y h_z \tau}$, which will correspond to separate differences

$$\begin{aligned}
&f_{h_x ijkp} = \frac{1}{h_x}(f_{i+1jkp} - f_{ijkp}) \quad f_{\overline{h_x} ijkp} = \frac{1}{h_x}(f_{ijkp} - f_{i-1jkp}) \\
&f_{h_y ijkp} = \frac{1}{h_y}(f_{ij+1kp} - f_{ijkp}) \quad f_{\overline{h_y} ijkp} = \frac{1}{h_y}(f_{ijkp} - f_{ij-1kp}) \\
&f_{h_z ijkp} = \frac{1}{h_z}(f_{ijk+1p} - f_{ijkp}) \quad f_{\overline{h_z} ijkp} = \frac{1}{h_z}(f_{ijkp} - f_{ijk-1p}) \\
&f_{h_x \overline{h_x} ijkp} = \frac{1}{h_x}(f_{h_x ijkp} - f_{\overline{h_x} ijkp}) = \frac{1}{h_x^2}(f_{i+1jkp} - 2f_{ijkp} + f_{i-1jkp}) \\
&f_{h_y \overline{h_y} ijkp} = \frac{1}{h_y}(f_{h_y ijkp} - f_{\overline{h_y} ijkp}) = \frac{1}{h_y^2}(f_{ij+1kp} - 2f_{ijkp} + f_{ij-1kp}) \\
&f_{h_z \overline{h_z} ijkp} = \frac{1}{h_z}(f_{h_z ijkp} - f_{\overline{h_z} ijkp}) = \frac{1}{h_z^2}(f_{ijk+1p} - 2f_{ijkp} + f_{ijk-1p}) \\
&f_{\bar{t} ijkp} = \frac{1}{\tau}(f_{ijkp} - f_{ijkp-1})
\end{aligned} \tag{20}$$

We rewrite items (3)-(5) taking into account the discretization of the original problem. The grid function $f_{h_x h_y h_z \tau} = f_{h_x h_y h_z \tau}\left(u_{h_x h_y h_z \tau}\right)$ will be the difference analogue of the function $f(x,y,z,y;u)$. Also, the function $u_{h_x h_y h_z \tau} = \{u_{ijkp}: i = \overline{1..X_h - 1}, j = \overline{1..Y_h - 1}, k = \overline{1..Z_h - 1}, p = \overline{1..M}\}$, which belongs to:

$$U_{h_x h_y h_z \tau} = \left\{ u_{ijkp}: \sum_{p=1}^{M} \sum_{i=1}^{X_h - 1} \sum_{j=1}^{Y_h - 1} \sum_{k=1}^{Z_h - 1} h_x h_y h_z \tau u_{ijkp}^2 \leq R^2 \right\} \tag{21}$$

Will be the difference analogue for the control $u(x,y,z,t)$. Then the criterion (4) for the minimization problem taking into account the function $f_{h_x h_y h_z \tau}$ takes the following form:

$$J_{h_x h_y h_z \tau}\left(u_{h_x h_y h_z \tau}\right) = \sum_{i=1}^{X_h - 1} \sum_{j=1}^{Y_h - 1} \sum_{k=1}^{Z_h - 1} h_x h_y h_z \left|f_{ijkP} - b_{ijk}\right|^2 \to inf, \tag{22}$$

$u_{h_x h_y h_z \tau} \in U_{h_x h_y h_z \tau}$,

Equation (6) taking into account (16) and boundary conditions:

$$\begin{gathered}
f_{\bar{t}ijkp} = \left(f_{h_x \overline{h_x} ijkp} + f_{h_y \overline{h_y} ijkp} + f_{h_z \overline{h_z} ijkp}\right) + u_{ijkp}, \\
i = \overline{1..X_h - 1}, j = \overline{1..Y_h - 1}, k = \overline{1..Z_h - 1}, p = \overline{0..P}; \\
f_{\overline{h_x} 1jkp} = f_{\overline{h_x} X_h jkp} = f_{\overline{h_y} i1kp} = f_{\overline{h_y} iY_h kp} = f_{\overline{h_z} ij1p} = f_{\overline{h_z} ijZ_h p} = 0, \\
p = \overline{1..P}; \\
f_{ijk0} = 0, i = \overline{0..X_h}, j = \overline{0..Y_h}, k = \overline{0..Z_h}
\end{gathered} \tag{23}$$

5. Theoretical Information

In the course of performing mathematical analysis, a number of theorems, equations, and inequalities were used that play a fundamental role or are often used in mathematical calculations and simplifications.

Partial Summation Formula;

$$\begin{gathered}
\sum_{n=p}^{q} a_n b_n = -\sum_{n=p}^{q-1} A_n (b_{n+1} - b_n) + A_q b_q - A_{p-1} b_p, \\
A_n = \sum_{k=0}^{n} a_k \text{, at } n \geq 0
\end{gathered} \tag{24}$$

Cauchy-Bunyakovsky inequality for sums and integrals;

$$\begin{gathered}
\sum_{i=1}^{n} |x_n * y_n| \leq \left(\sum_{i=1}^{n} |x_n|^2\right)^{\frac{1}{2}} * \left(\sum_{i=1}^{n} |y_n|^2\right)^{\frac{1}{2}} \\
\int_a^b f(x) g(x) dx \leq \left(\int_a^b f^2(x) dx\right)^{\frac{1}{2}} \left(\int_a^b g^2(x) dx\right)^{\frac{1}{2}}
\end{gathered} \tag{25}$$

Lemma 1. [1] If some quantities $\varphi_i, i = 0, \dots, N$ satisfy the inequalities:

$$0 \leq \varphi_0 \leq a, 0 \leq \varphi_{i+1} \leq a + b \sum_{m=0}^{i} \varphi_m \text{, } i = 1, \dots, N-1, b \geq 0,$$

Then the estimate $0 \leq \varphi_i \leq a(1+b)^i$ is fair, at $i = 0, \dots, N$. If

$$0 \leq \varphi_{i-1} \leq a + b \sum_{m=i}^{N-1} \varphi_m \text{, } i = 1, \dots, N-1, 0 \leq \varphi_{N-1} \leq a,$$

Then the estimate $0 \leq \varphi_{i-1} \leq a(1+b)^{N-i-1}$ is fair, at $i = 0, \dots, N-1$.

Elementary Inequalities:

$$\begin{gathered}
|ab| \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2, (a+b)^2 \leq 2a^2 + 2b^2, \\
(a+b+c)^2 \leq 3(a^2 + b^2 + c^2) \ \forall a, b, c \in \mathbb{R} \ \forall \varepsilon > 0.
\end{gathered} \tag{26}$$

6. Analysis of the Differential Problem

We begin the analysis of the differential problem by deriving two estimates for sufficiently smooth classical solutions to problem (5), (6), which will be emphasized in future work. Further actions are based on functional inequalities, which are sufficiently developed in Vasilev (2002) study [17].

We multiply equation (1.6) by $f(x, y, z, t; u)$ and integrate the resulting equality over the rectangle

$Q_\tau = \{(x, y, z, t): 0 \le x \le l_x, 0 \le y \le l_y, 0 \le z \le l_z, 0 \le t \le \tau\}$,

Where $\tau -$ arbitrary fixed point in time, $0 \le \tau \le T$.

$$\int_{Q_\tau} \frac{\partial f}{\partial t} f dxdydzdt - \int_{Q_\tau} \Delta f \cdot f dxdydzdt = \int_{Q_\tau} uf dxdydzdt \tag{27}$$

In view of conditions (2), we transform the first term from the left-hand side:

$$\int_{Q_\tau} \frac{\partial f}{\partial t} f Q_\tau = \int_{Q_3} \frac{1}{2}\left(\int_0^\tau \frac{\partial}{\partial t}(f^2) dt\right) dQ_3 = \frac{1}{2}\int_{Q_3} f^2(x, y, z, \tau) dQ_3 \tag{28}$$

To estimate the second term, we introduce each term of the Laplace operator under the differential sign, after which we apply the boundary conditions (5). As a result, we have:

$$\int_{Q_\tau} \Delta f \cdot f dQ_\tau = -\int_{Q_\tau} \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right) dQ_\tau \tag{29}$$

We use the Cauchy-Bunyakovsky formula (21) for the right-hand side of equality (27), after which we pass to the maximum in time for the classical solution of problem (5), (6). We have:

$$\int_{Q_\tau} uf dQ_\tau \le \int_0^\tau \left(\int_{Q_3} u^2 dQ_3\right)^{\frac{1}{2}} \left(\int_{Q_3} f^2 dQ_3\right)^{\frac{1}{2}} dt \le$$

$$\le \max_{0 \le t \le \tau} \left(\int_{Q_3} f^2 dQ_3\right)^{\frac{1}{2}} \int_0^\tau \left(\int_{Q_3} u^2 dQ_3\right)^{\frac{1}{2}} dt \le \max_{0 \le t \le \tau} \left(\int_{Q_\tau} f^2 dQ_\tau\right)^{\frac{1}{2}} \sqrt{T} \|u\|_{L_2(Q)}$$

$$\int_{Q_\tau} uf dQ_\tau \le \max_{0 \le t \le \tau} \left(\int_{Q_\tau} f^2 dQ_\tau\right)^{\frac{1}{2}} \sqrt{T} \|u\|_{L_2(Q)} \tag{30}$$

We replace the terms in Equation (27) in accordance with formulas (28), (29) and (30), we have:

$$\frac{1}{2}\int_{Q_3} f^2(x, y, z, \tau) dQ_3 + \int_{Q_\tau} \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right) dQ_\tau \le$$
$$\le \max_{0 \le t \le \tau} \left(\int_{Q_\tau} f^2 dQ_\tau\right)^{\frac{1}{2}} \sqrt{T} \|u\|_{L_2(Q)} \tag{31}$$

Let us estimate this inequality. To do this, we remove each term from the right-hand side in turn. Based on this, we evaluate the first term.

$$\int_{Q_3} f^2(x, y, z, \tau) dQ_3 \le \max_{0 \le t \le T} \left(\int_{Q_\tau} f^2 dQ_\tau\right)^{\frac{1}{2}} 2\sqrt{T} \|u\|_{L_2(Q)} \ \forall \tau \in [0, T]$$

Therefore, if we take the integral of the square of the function f with respect to the maximum τ on the interval $[0, T]$, square and extract the square root, and then use the estimate for the first term, we obtain the following inequality:

$$\max_{0 \le \tau \le T} \int_{Q_3} f^2(x, y, z, \tau) dQ_3 \le \max_{0 \le t \le T} \left(\int_{Q_3} f^2 dQ_\tau\right)^{\frac{1}{2}} 2\sqrt{T} \|u\|_{L_2(Q)}$$

Or:

$$\max_{0\le t\le T}\int\limits_{Q_3} f^2(x,y,z,t)dQ_3 \le 4T\|u\|^2{}_{L_2(Q)} \tag{32}$$

From (31) we make an estimate for the second term, taking into account the estimate (32), we have:

$$\int\limits_{Q_\tau}\left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2\right)dQ_\tau \le \max_{0\le t\le\tau}\left(\int\limits_{Q_\tau} f^2 dQ_\tau\right)^{\frac{1}{2}}\sqrt{T}\|u\|_{L_2(Q)} \le$$
$$\le 2T\|u\|^2{}_{L_2(Q)} \tag{33}$$

Based on inequality (31) and estimates (32) and (33), we obtain the first estimate for a sufficiently smooth solution to problem (31) and (32):

$$\max_{0\le t\le T}\int\limits_{Q_3} f^2(x,y,z,\tau)dQ_3+\int\limits_{Q_\tau}\left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2\right)dQ_\tau \le 6T\|u\|^2{}_{L_2(Q)} \tag{34}$$

Multiply equation (32) by $\frac{\partial f}{\partial t}$ and integrate over the domain Q_τ:

$$\int\limits_{Q_\tau}\left(\frac{\partial f}{\partial t}\right)^2 dQ_\tau = \int\limits_{Q_\tau}\Delta f\cdot\frac{\partial f}{\partial t}dQ_\tau+\int\limits_{Q_\tau} u\frac{\partial f}{\partial t}dQ_\tau \tag{35}$$

We estimate the first scalar product from the right-hand side. To do this, we introduce each term of the Laplace operator under the differential sign. As a result, we get:

$$\int\limits_{Q_\tau}\Delta f\cdot\frac{\partial f}{\partial t}dQ_\tau=\int\limits_0^\tau\left(\int\limits_0^{l_y}\int\limits_0^{l_z}\frac{\partial f}{\partial t}\frac{\partial f}{\partial x}\bigg|_0^{l_x}dydz+\int\limits_0^{l_x}\int\limits_0^{l_z}\frac{\partial f}{\partial t}\frac{\partial f}{\partial y}\bigg|_0^{l_y}dxdz+\right.$$
$$\left.+\int\limits_0^{l_x}\int\limits_0^{l_y}\frac{\partial f}{\partial t}\frac{\partial f}{\partial z}\bigg|_0^{l_z}dxdy-\int\limits_{Q_3}\left(\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial t\partial x}+\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial t\partial y}+\frac{\partial f}{\partial z}\frac{\partial^2 f}{\partial t\partial z}\right)dQ_3\right)dt=$$

Taking into account the boundary conditions (5), only the last integral does not vanish. If we introduce the derivative of the function with respect to each variable under the differential sign and use the main theorem of mathematical analysis, we get:

$$=\int\limits_{Q_3}\left(\int\limits_0^\tau\frac{1}{2}\frac{\partial}{\partial t}\left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2\right)dt\right)dQ_3$$
$$=-\frac{1}{2}\left(\left(\frac{\partial f(x,y,z,\tau)}{\partial x}\right)^2+\left(\frac{\partial f(x,y,z,\tau)}{\partial y}\right)^2+\left(\frac{\partial f(x,y,z,\tau)}{\partial z}\right)^2\right)dQ_3$$

As a result, we obtain the following equality:

$$\int\limits_{Q_\tau}\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial f}{\partial t}+\frac{\partial^2 f}{\partial y^2}\frac{\partial f}{\partial t}+\frac{\partial^2 f}{\partial z^2}\frac{\partial f}{\partial t}\right)dQ_\tau$$
$$=-\frac{1}{2}\int\limits_{Q_3}\left(\left(\frac{\partial f(x,y,z,\tau)}{\partial x}\right)^2+\left(\frac{\partial f(x,y,z,\tau)}{\partial y}\right)^2+\left(\frac{\partial f(x,y,z,\tau)}{\partial z}\right)^2\right)dQ_3 \tag{36}$$

Based on formula (36) and the elementary inequality of paragraph 1.4 for the product from formula (35), we have;

$$\frac{1}{2}\int\limits_{Q_3}\left(\left(\frac{\partial f(x,y,z,\tau)}{\partial x}\right)^2+\left(\frac{\partial f(x,y,z,\tau)}{\partial y}\right)^2+\left(\frac{\partial f(x,y,z,\tau)}{\partial z}\right)^2\right)dQ_3+$$

$$+\int\limits_{Q_\tau}\left(\frac{\partial f}{\partial t}\right)^2 dQ_\tau = \int\limits_{Q_\tau} u\frac{\partial f}{\partial t}Q_\tau \le \frac{1}{2}\|u\|^2_{L_2(Q)} + \frac{1}{2}\int\limits_{Q_\tau}\left(\frac{\partial f}{\partial t}\right)^{\frac{1}{2}} Q_\tau$$

Or:

$$\int\limits_{Q_\tau}\left(\frac{\partial f}{\partial t}\right)^2 dQ_\tau +$$

$$+\int\limits_{Q_3}\left(\left(\frac{\partial f(x,y,z,\tau)}{\partial x}\right)^2 + \left(\frac{\partial f(x,y,z,\tau)}{\partial y}\right)^2 + \left(\frac{\partial f(x,y,z,\tau)}{\partial z}\right)^2\right) dQ_3 \le \|u\|^2_{L_2(Q)}$$

$\forall\tau \in [0,T]$

Hence we have 2 inequalities:

$$\int\limits_{Q_\tau}\left(\frac{\partial f}{\partial t}\right)^2 dQ_\tau \le \|u\|^2_{L_2(Q)},$$

$$\int\limits_{Q_3}\left(\left(\frac{\partial f(x,y,z,\tau)}{\partial x}\right)^2 + \left(\frac{\partial f(x,y,z,\tau)}{\partial y}\right)^2 + \left(\frac{\partial f(x,y,z,\tau)}{\partial z}\right)^2\right) dQ_3 \le \|u\|^2_{L_2(Q)}$$

$\forall\tau \in [0,T]$.

We use the fact that τ takes any value on the interval $[0,T]$, we get:

$$\int\limits_{Q}\left(\frac{\partial f}{\partial t}\right)^2 dQ \le \|u\|^2_{L_2(Q)},$$

$$\max_{0\le\tau\le T}\int\limits_{Q_3}\left(\left(\frac{\partial f(x,y,z,\tau)}{\partial x}\right)^2 + \left(\frac{\partial f(x,y,z,\tau)}{\partial y}\right)^2 + \le\right. \tag{37}$$

$$\left.+\left(\frac{\partial f(x,y,z,\tau)}{\partial z}\right)^2\right) dQ_3 \le \|u\|^2_{L_2(Q)}\ \forall\tau \in [0,T].$$

In addition, if we integrate equation (6) over the domain Q taking into account (37), we have:

$$\int\limits_{Q}(\Delta f)^2 dQ = \int\limits_{Q}\left(\frac{\partial f}{\partial t} - u\right)^2 dQ \le 2\int\limits_{Q}\left(\frac{\partial f}{\partial t}\right)^2 dQ + 2\int\limits_{Q} u^2 dQ \tag{38}$$

$$\le 4\|u\|^2_{L_2(Q)}$$

Adding inequalities (37) and (38) we obtain the second estimate for a sufficiently smooth solution:

$$\max_{0\le\tau\le T}\int\limits_{Q_3}\left(\left(\frac{\partial f(x,y,z,\tau)}{\partial x}\right)^2 + \left(\frac{\partial f(x,y,z,\tau)}{\partial y}\right)^2 + \left(\frac{\partial f(x,y,z,\tau)}{\partial z}\right)^2\right) dQ_3 +$$

$$+\int\limits_{Q}\left(\frac{\partial f}{\partial t}\right)^2 dQ + \int\limits_{Q}(\Delta f)^2 dQ \le 6\|u\|^2_{L_2(Q)} \tag{39}$$

We use the Friedrichs inequality and inequalities (37), and also taking the maximum in time for differentials with respect to spatial variables, we estimate the square of the solution with respect to the control function:

$$4\int\limits_{Q} f^2 dQ \le \frac{l^2_{\max}}{2}\max_{0\le t\le T}\int\limits_{Q_3}\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right) dQ_3 + \frac{T^2}{2}\int\limits_{Q}\left(\frac{\partial f}{\partial x}\right)^2 dQ \le$$

$$\leq \frac{l_{\max}^2}{2}\|u\|_{L_2(Q)}^2 + \frac{T^2}{2}\|u\|_{L_2(Q)}^2,$$

$$\forall(x,y,z,t) \in Q, l_{\max} = \max\{l_x, l_y, l_z\}.$$

From here we get the energy estimate:

$$\max_{(x,y,z,t)\in\bar{Q}} \int_Q f^2 dQ \leq C\|u\|_{L_2(Q)}^2,$$
$$C = \frac{(l_{\max}^2 + T^2)}{8}, l_{\max} = \max\{l_x, l_y, l_z\} \tag{40}$$

7. Analysis of a Discrete Task

Using analogues with estimates (34) and (39), we derive the corresponding estimates for the discrete problem. We multiply Equation (19) by $h_x h_y h_z \tau f_{ijkp} = h\tau f_{ijkp}$ and sum over i, j, k from 1 to $X_h - 1 = \overline{X_h}$, from 1 to $Y_h - 1 = \overline{Y_h}$, from 1 to $Z_h - 1 = \overline{Z_h}$ respectively:

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{\bar{t}ijkp} f_{ijkp} - \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau\left(f_{\bar{x}xijkp} + f_{\bar{y}yijkp} + f_{\bar{z}zijkp}\right) f_{ijkp} =$$
$$= \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp} f_{ijkp}\, , p = 1, \dots, P \tag{41}$$

It is easy to verify that;

$$\tau f_{\bar{t}ijkp} f_{ijkp} \geq \frac{1}{2}\left(f_{ijkp}^2 - f_{ijkp-1}^2\right),$$
$$i = \overline{1..X_h - 1}, j = \overline{1..Y_h - 1}, k = \overline{1..Z_h - 1}, p = \overline{1..M} \tag{42}$$

From here;

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{\bar{t}ijkp} f_{ijkp} \geq \frac{1}{2}\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h f_{ijkp}^2 - \frac{1}{2}\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h f_{ijkp-1}^2\, , p = \overline{1..P} \tag{43}$$

In order to transform the second term from the left side of Equation (41), we use the summation formula by parts (20) and the boundary conditions (19):

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{\bar{x}xijkp} f_{ijkp} = -\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{\bar{x}ijkp}^2 \tag{44}$$

Similarly, the formula is applicable to the spatial variables$y$ and z.

We substitute formulas (43) and (44) in the formula (41):

$$\frac{1}{2}\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} \left(h f_{ijkp}^2 - h f_{ijkp-1}^2\right) + \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau\left(f_{\bar{x}ijkp}^2 + f_{\bar{y}ijkp}^2 + f_{\bar{z}ijkp}^2\right) \leq$$
$$\leq \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp} f_{ijkp}\, , p = \overline{1..P} \tag{45}$$

Inequality (45) is summed over p from 1 to some $\wp$, where $\wp$ on the interval $1 \leq \wp \leq P$. We use the boundary condition $f_{ijk0} = 0, i = \overline{0..X_h}, j = \overline{0..Y_h}, k = \overline{0..Z_h}$. Then, if we expand the right-hand side of inequality (45) according to the Cauchy-Bunyakovsky formula (21) and make the maximum transition in time for a discrete solution, we obtain a difference analogue of the inequality (30):

$$\frac{1}{2}\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} hf_{ijk\wp}^2 + \sum_{p,i,j,k=1}^{\wp,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp} f_{ijkp} \le \max_{1\le p\le P}\left(\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} hf_{ijkp}^2\right)^{\frac{1}{2}} \cdot \sqrt{T}\left(\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp}^2\right)^{\frac{1}{2}} \forall\wp, 1\le\wp\le P. \tag{46}$$

Then, if we carry out mathematical transformations similar to those carried out when estimating inequality (30), then we obtain the following estimates for the left and right terms

$$\max_{1\le p\le P}\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} hf_{ijkp}^2 \le 4T\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp}^2 \tag{47}$$

$$\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau\left(f_{\bar{x}ijkp}^2 + f_{\bar{y}ijkp}^2 + f_{\bar{z}ijkp}^2\right) \le 2T\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp}^2 \tag{48}$$

If we add inequalities (47) and (48), we obtain a difference estimate similar to the integral estimate (34) up to a constant:

$$\max_{1\le p\le P}\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} hf_{ijk\wp}^2 + \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau\left(f_{\bar{x}ijkp}^2 + f_{\bar{y}ijkp}^2 + f_{\bar{z}ijkp}^2\right) \le \le 6T\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp}^2 \tag{49}$$

Find the difference analogue for the estimate (39). To do this, we multiply the equation from (19) by $h\tau f_{\bar{t}ijkp}$ and summarize the resulting expression by i, j, k by $1 \le i \le X_h - 1; 1 \le j \le Y_h - 1; 1 \le k \le Z_h - 1$.

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{\bar{t}ijkp}^2 - \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau\left(f_{\bar{x}xijkp} + f_{\bar{y}yijkp} + f_{\bar{z}zijkp}\right)f_{\bar{t}ijkp} = = \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp} f_{\bar{t}ijkp}, p = 1,\dots,P \tag{50}$$

We use the summation formula in parts (20) in accordance with formula (44) and the boundary conditions (19) to estimate the second term from the left-hand side. We get:

$$-\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau\left(f_{\bar{x}xijkp} + f_{\bar{y}yijkp} + f_{\bar{z}zijkp}\right)f_{\bar{t}ijkp} = = \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau\left(f_{\bar{x}ijkp} f_{\bar{t}\bar{x}ijkp} + f_{\bar{y}ijkp} f_{\bar{t}\bar{y}ijkp} + f_{\bar{z}ijkp} f_{\bar{t}\bar{z}ijkp}\right), p = 1,\dots,P \tag{51}$$

We use formula (42) to estimate the right-hand side of the equality (51):

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau\left(f_{\bar{x}ijkp} f_{\bar{t}\bar{x}ijkp} + f_{\bar{y}ijkp} f_{\bar{t}\bar{y}ijkp} + f_{\bar{z}ijkp} f_{\bar{t}\bar{z}ijkp}\right) \ge \frac{1}{2}\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\left(f_{\bar{x}ijkp}^2 + f_{\bar{y}ijkp}^2 + f_{\bar{z}ijkp}^2 - f_{\bar{x}ijkp-1}^2 - f_{\bar{y}ijkp-1}^2 - f_{\bar{z}ijkp-1}^2\right) \tag{52}$$

Substitute this estimate in (50). We have:

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{\bar{t}ijkp}^2 - \frac{1}{2}\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\left(f_{\bar{x}ijkp}^2 + f_{\bar{y}ijkp}^2 + f_{\bar{z}ijkp}^2\right) +$$
$$+\frac{1}{2}\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\left(f_{\bar{x}ijkp-1}^2 + f_{\bar{y}ijkp-1}^2 + f_{\bar{z}ijkp-1}^2\right) \le \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp} f_{\bar{t}ijkp}, \tag{53}$$
$$p = 1, \dots, P$$

The left side of inequality (53) is summed over p from 1 to some $\wp$, where $\wp$ is in the interval $1 \le \wp \le P$. Given $f_{ijk0} = 0, i = \overline{0..X_h}, j = \overline{0..Y_h}, k = \overline{0..Z_h}$, we obtain:

$$\sum_{p,i,j,k=1}^{\wp,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{\bar{t}ijkp}^2 + \frac{1}{2}\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\left(f_{\bar{x}ijk\wp}^2 + f_{\bar{y}ijk\wp}^2 + f_{\bar{z}ijk\wp}^2\right) \le$$
$$\le \sum_{p,i,j,k=1}^{\wp,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp} f_{\bar{t}ijkp} \le \frac{1}{2}\sum_{p,i,j,k=1}^{\wp,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{\bar{t}ijkp}^2 + \frac{1}{2}\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp}^2$$

Or, if we transfer the first amount from the right to the left:

$$\sum_{p,i,j,k=1}^{\wp,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{\bar{t}ijkp}^2 + \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\left(f_{\bar{x}ijk\wp}^2 + f_{\bar{y}ijk\wp}^2 + f_{\bar{z}ijk\wp}^2\right) \le$$
$$\le \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp}^2, \forall \wp, 1 \le \wp \le P \tag{54}$$

From inequality(54) we can obtain two corollaries:

$$\max_{1\le\wp\le P} \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\left(f_{\bar{x}ijk\wp}^2 + f_{\bar{y}ijk\wp}^2 + f_{\bar{z}ijk\wp}^2\right) \le \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp}^2 \tag{55}$$

$$\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{\bar{t}ijkp}^2 \le \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp}^2 \tag{56}$$

Finally, by squaring Equation (19) we apply the elementary inequality for the square of the sum from (4) and estimate the result with (56):

$$\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{\bar{t}ijkp}^2 = \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau\left(f_{\bar{x}xijkp} + f_{\bar{y}yijkp} + f_{\bar{z}zijkp} - u_{ijkp}\right)^2 \le$$
$$\le 4\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp}^2 \tag{57}$$

If we add inequalities (55)-(57), then we get the difference analogue of estimate (39):

$$\max_{1\le p\le P} \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\left(f_{\bar{x}ijkp}^2 + f_{\bar{y}ijkp}^2 + f_{\bar{z}ijkp}^2\right) + \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{\bar{t}ijkp}^2 +$$
$$+\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{\bar{t}ijkp} \le 6\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau u_{ijkp}^2 \tag{58}$$

8. Evaluation of the Difference of Differential Decision and Discrete Analogue

We introduce the Hilbert space $L_{2h_xh_yh_z\tau} = L_{2h\tau}$, which is the difference analogue of the space $L_2(Q)$. The elements of this space will be the grid functions $f_{h_xh_yh_z\tau} = f_{h\tau} = \{f_{ijkp}, i = \overline{1..\overline{X_h}}, j = \overline{1..\overline{Y_h}}, k = \overline{1..\overline{Z_h}}, p = \overline{1..P}\}$, and the scalar and vector spaces are defined as follows:

$$\langle f_{h\tau}, g_{h\tau}\rangle = \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{ijkp} g_{ijkp},$$
$$\|f_{h\tau}\|_{L_{2h\tau}} = \left(\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{ijkp}^2\right)^{\frac{1}{2}} \tag{59}$$

By $b_{h\tau}f_{h\tau}$ we denote the piecewise constant continuation of the grid function $f_{h\tau}$ according to the rule;

$$\begin{aligned} & b_{h\tau}f_{h\tau} = (b_{h\tau}f_{h\tau})(x,y,z,t) = f_{ijkp} \\ & (x,y,z,t) \in Q_{ijkp} = \{(x,y,z,t): x_i \le x \le x_{i+1}, y_j \le y \le y_{j+1}, z_k \le z \\ & \le z_{k+1}, t_{p-1} \le t \le t_p\}, (x,y,z) \in Q_{ijk} = \{(x,y,z): x_i \le x \le x_{i+1}, \\ & y_j \le y \le y_{j+1}, z_k \le z \le z_{k+1}\} \\ & , i = \overline{1..\overline{X_h}}, j = \overline{1..\overline{Y_h}}, k = \overline{1..\overline{Z_h}}, p = \overline{1..P}; \end{aligned} \tag{60}$$

The domain of the function $b_{h\tau}f_{h\tau}$ is denoted by $Q_h = \{(x,y,z,t)\text{: } h_x \le x \le l_x, h_y \le y \le l_y, h_z \le z \le l_z, 0 \le t \le T\}$. We note that:

$$\int_{Q_h} b_{h\tau}f_{h\tau} dQ_h = \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{ijkp} \quad \|b_{h\tau}f_{h\tau}\|_{L_{2(Q_h)}} = \|f_{h\tau}\|_{L_{2h\tau}}$$
$$\langle b_{h\tau}f_{h\tau}, b_{h\tau}g_{h\tau}\rangle_{L_{2(Q_h)}} = \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} h\tau f_{ijkp} g_{ijkp} = \langle f_{h\tau}, g_{h\tau}\rangle_{L_{2h\tau}}, \tag{61}$$

Based on (60), (61), we rewrite the difference equation (16):

$$b_{h\tau}f_{\bar{t}h\tau} - b_{h\tau}\left(f_{\bar{x}xh\tau} + f_{\bar{y}yh\tau} + f_{\bar{z}zh\tau}\right) = b_h u_{h\tau}, (x,y,z,t) \in Q_h \tag{62}$$

Subtract (62) from equation (6), п multiply the resulting equality by $f - b_{h\tau}f_{h\tau}$ and integrate over the domain Q_h:

$$\begin{aligned} & \int_{Q_h} \left(\frac{\partial f}{\partial t} - b_{h\tau}f_{\bar{t}h\tau}\right)(f - b_{h\tau}f_{h\tau})dQ_h - \\ & - \int_{Q_h} \left(\Delta f - b_{h\tau}\left(f_{\bar{x}xh\tau} + f_{\bar{y}yh\tau} + f_{\bar{z}zh\tau}\right)\right)(f - b_{h\tau}f_{h\tau})dQ_h = \\ & = \int_{Q_h} (u - b_{h\tau}u_{h\tau})(f - b_{h\tau}f_{h\tau})dQ_h \end{aligned} \tag{63}$$

We estimate the first term from the left side of the equality (63). We replace the integration over the entire domain with summation in accordance with the formula (61):

$$\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijkp}} \left(\frac{\partial f}{\partial t} - b_{h\tau}f_{\bar{t}h\tau}\right)(f - b_{h\tau}f_{h\tau})dQ_{ijkp} =$$
$$= \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijkp}} \left(\frac{1}{2}\frac{\partial}{\partial t}\left(f - (t - t_p)f_{\bar{t}ijkp} - f_{ijkp}\right)^2 + \right.$$

$$+\left(\frac{\partial f}{\partial t}-f_{\bar{t}ijkp}\right)\left(t-t_p\right)f_{\bar{t}ijkp}\right)dQ_{ijkp}=$$

For the first term, we substitute the limiting value for integration over the time variable, and we open the first bracket for the second:

$$=\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\frac{1}{2}\sum_{p=1}^{P}\left(\left(f\left(x,y,z,t_p\right)-f_{ijkp}\right)^2-\left(f\left(x,y,z,t_{\bar{p}}\right)-f_{ijk\bar{p}}\right)^2\right)dQ_{ijk}+$$

$$\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijkp}}\left(f\left(t-t_p\right)f_{\bar{t}ijkp}\Big|_{t_{\bar{p}}}^{t_p}-\int_{t_{\bar{p}}}^{t_p}ff_{\bar{t}ijkp}dt-f_{\bar{t}ijkp}^2\int_{t_{\bar{p}}}^{t_p}\left(t-t_p\right)dt\right)dQ_{ijkp}$$

For the first sum, we go through the cycle in the time variable, opening the squares of the difference and using the boundary condition $f_{ijk0}=0$, and for the second, we calculate the time integral for the third term, substitute the limit values in the first and third elements of the term bracket. Then the final inequality takes the following form:

$$\int_{Q_h}\left(\frac{\partial f}{\partial t}-b_{h\tau}f_{\bar{t}h\tau}\right)(f-b_{h\tau}f_{\bar{t}h\tau})dQ_h\geq$$
$$\geq\frac{1}{2}\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left|f(x,y,z,t_P)-f_{ijkP}\right|^2dQ_{ijk}+ \tag{64}$$
$$+\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijkp}}\left(f\left(x,y,z,t_{p-1}\right)-f\right)f_{\bar{t}ijkp}dQ_{ijkp}$$

We transform the second term from the left-hand side of (64). We note that;

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left(\left(\frac{\partial^2 f}{\partial x^2}-f_{\bar{x}x}(x_i,y,z,t)\right)\left(f-f_{ijkp}\right)\right)dQ_{ijk}=$$
$$=\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left(\left(\frac{\partial^2 f}{\partial x^2}-f_{\bar{x}x}(x_i,y,z,t)\right)\left(f-f_{ijkp}\right)\right)dQ_{ijk}+$$
$$+\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left(\left(f_{\bar{x}x}(x_i,y,z,t)-f_{\bar{x}xijkp}\right)\left(f-f(x_i,y,z,t)\right)\right)dQ_{ijk}+ \tag{65}$$
$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left(f_{\bar{x}x}(x_i,y,z,t)-f_{\bar{x}xijkp}\right)\left(f(x_i,y,z,t)-f_{ijkp}\right)dQ_{ijk}$$
$$\forall t,t_{p-1}<t\leq t_p,p=1,\dots,P$$

We transform the first term from the right-hand side of (65). To do this, we take out the differential from the first bracket, we apply integration by parts. For the part of the expression in which the limit values are substituted, we will go through the cycle in the variable x. Then, having completed the mathematical operations, we arrive at the following inequality:

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}-(x-x_i)f_{\bar{x}x}(x_i,y,z,t)-f_{\bar{x}}(x_i,y,z,t)\right)\cdot$$
$$\cdot\left(f-f_{ijkp}\right)dQ_{ijk}=\sum_{j,k=1}^{\overline{Y_h},\overline{Z_h}}\int_{Q_{jk}}\left(\left(\frac{\partial f(x_{X_h},y,z,t)}{\partial x}-f_{\bar{x}}\left(x_{X_h},y,z,t\right)\right)\cdot\right. \tag{66}$$

$$\cdot\left(f\left(x_{X_h},y,z,t\right)-f_{X_hjkp}\right)-\left(\frac{\partial f(x_1,y,z,t)}{\partial x}-f_{\bar{x}}(x_1,y,z,t)\right)\cdot$$

$$\cdot\left(f(x_1,y,z,t)-f_{1jkp}\right)\Big)dQ_{jk}+$$

$$+\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{jk}}\left(\frac{\partial f(x_{i+1},y,z,t)}{\partial x}-f_{\bar{x}}(x_{i+1},y,z,t)\right)h_X f_{xijkp}dQ_{jk}-$$

$$-\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left(\frac{\partial f(x_i,y,z,t)}{\partial x}-f_{\bar{x}}(x_i,y,z,t)-(x-x_i)\cdot\right.$$

$$\left.\cdot f_{\bar{x}x}(x_i,y,z,t)\right)\frac{\partial f}{\partial x}dQ_{ijk}$$

$$\forall t,t_{p-1}<t\le t_p,p=\overline{1..P}$$

The third term from the right-hand side of (65), and using the formula for summing by parts (17), can be represented as follows:

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left(\left(f_{\bar{x}x}(x_i,y,z,t)-f_{\bar{x}xijkp}\right)\left(f(x_i,y,z,t)-f_{ijkp}\right)\right)dQ_{ijk}=$$

$$=\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{jk}}h_X\left(f(x_i,y,z,t)-f_{ijkp}\right)_{\bar{x}x}\left(f(x_i,y,z,t)-f_{ijkp}\right)dQ_{jk}=$$

$$=\sum_{j,k=1}^{\overline{Y_h},\overline{Z_h}}\int_{Q_{jk}}\left(-\sum_{i=1}^{\overline{X_h}}h_X\left(f_{\bar{x}}(x_i,y,z,t)-f_{\bar{x}ijkp}\right)^2+\right. \tag{67}$$

$$+\left(f_{\bar{x}}\left(x_{X_h},y,z,t\right)-f_{\bar{x}X_hjkp}\right)\cdot$$

$$\cdot\left(f\left(x_{\overline{X_h}},y,z,t\right)-f_{\overline{X_h}jkp}\right)-\left(f_{\bar{x}}(x_1,y,z,t)-f_{\bar{x}1jkp}\right)\cdot$$

$$\left.\cdot\left(f(0,y,z,t)-f_{0jkp}\right)\right)dQ_{jk},\forall t,t_{p-1}<t\le t_p,p=\overline{1..P}$$

Performing similar mathematical operations (65)-(67), we can obtain estimates for the variables y and z, replacing the variable x with another spatial variable, taking into account the limits of summation and integration.

We substitute the obtained inequality (64) and equality (65) taking into account (66) and (67) into (61). We get:

$$\frac{1}{2}\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left|f(x,y,z,t_P)-f_{ijkP}\right|^2dQ_{ijk}+$$

$$+\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijkp}}\left(\left(f(x_i,y,z,t)-f_{\bar{x}ijkp}\right)^2+\left(f\left(x,y_j,z,t\right)-f_{\bar{y}ijkp}\right)^2+\right. \tag{68}$$

$$\left.+\left(f(x,y,z_k,t)-f_{\bar{z}ijkp}\right)^2\right)dQ_{ijkp}\le\sum_{i=1}^{10}Fi$$

Where;

$$F_1=\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijkp}}\left(f-f\left(x,y,z,t_{p-1}\right)\right)f_{\bar{t}ijkp}dQ_{ijkp} \tag{69}$$

$$F_2=\sum_{p=1}^{P}\int_{t_{p-1}}^{t_p}\left(\sum_{j,k=1}^{\overline{Y_h},\overline{Z_h}}\int_{Q_{jk}}\left(\frac{\partial f(x_{X_h},y,z,t)}{\partial x}-f_{\bar{x}}(x_{X_h},y,z,t)\right)\cdot\right.$$
$$\cdot\left(f(x_{X_h},y,z,t)-f_{X_hjkp}\right)dQ_{jk}$$
$$+\sum_{i,k=1}^{\overline{X_h},\overline{Z_h}}\int_{Q_{ik}}\left(\frac{\partial f(x,y_{Y_h},z,t)}{\partial y}-f_{\bar{y}}(x,y_{Y_h},z,t)\right)\cdot\left(f(x,y_{Y_h},z,t)-\right. \tag{70}$$
$$\left.-f_{iY_hkp}\right)dQ_{ik}+\sum_{i,j=1}^{\overline{X_h},\overline{Y_h}}\int_{Q_{ij}}\left(\frac{\partial f(x,y,z_{Z_h},t)}{\partial z}-f_{\bar{z}}(x,y,z_{Z_h},t)\right)\cdot$$
$$\left.\cdot\left(f(x,y,z_{Z_h},t)-f_{ijZ_hp}\right)dQ_{ij}\right)dt$$

$$F_3=-\sum_{p=1}^{P}\int_{t_{p-1}}^{t_p}\left(\sum_{j,k=1}^{\overline{Y_h},\overline{Z_h}}\int_{Q_{jk}}\left(\frac{\partial f(x_1,y,z,t)}{\partial x}-f_{\bar{x}}(x_1,y,z,t)\right)\cdot\right.$$
$$\cdot\left(f(x_1,y,z,t)-f_{1jkp}\right)dQ_{jk}+\sum_{i,k=1}^{\overline{X_h},\overline{Z_h}}\int_{Q_{ik}}\left(f(x,y_1,z,t)-f_{i1kp}\right)\cdot$$
$$\cdot\left(\frac{\partial f(x,y_1,z,t)}{\partial y}-f_{\bar{y}}(x,y_1,z,t)\right)dQ_{ik}+\sum_{i,j=1}^{\overline{X_h},\overline{Y_h}}\int_{Q_{ij}}\left(f(x,y,z_1,t)-\right. \tag{71}$$
$$\left.\left.-f_{ij1p}\right)\left(\frac{\partial f(x,y,z_1,t)}{\partial z}-f_{\bar{z}}(x,y,z_1,t)\right)dQ_{ij}\right)dt$$

$$F_4=\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijkp}}\left(f_{xijkp}\left(\frac{\partial f(x_{i+1},y,z,t)}{\partial x}-f_{\bar{x}}(x_{i+1},y,z,t)\right)+\right.$$
$$+f_{yijkp}\left(\frac{\partial f(x,y_{j+1},z,t)}{\partial y}-f_{\bar{y}}(x,y_{j+1},z,t)\right)+ \tag{72}$$
$$\left.+f_{zijkp}\left(\frac{\partial f(x,y,z_{k+1},t)}{\partial z}-f_{\bar{z}}(x,y,z_{k+1},t)\right)\right)Q_{ijkp}$$

$$F_5=-\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijkp}}\left(\left(\frac{\partial f}{\partial x}-f_{\bar{x}}(x_i,y,z,t)\right)\frac{\partial f}{\partial x}+\right.$$
$$\left.\left(\frac{\partial f}{\partial y}-f_{\bar{y}}(x,y_j,z,t)\right)\frac{\partial f}{\partial y}+\left(\frac{\partial f}{\partial z}-f_{\bar{z}}(x,y,z_k,t)\right)\frac{\partial f}{\partial z}\right)dQ_{ijkp} \tag{73}$$

$$F_6=\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijkp}}\left(f_{\bar{x}x}(x_i,y,z,t)(x-x_i)\frac{\partial f}{\partial x}+\right.$$
$$\left.f_{\bar{y}y}(x,y_j,z,t)(y-y_j)\frac{\partial f}{\partial y}+f_{\bar{z}z}(x,y,z_k,t)(z-z_k)\frac{\partial f}{\partial z}\right)dQ_{ijkp} \tag{74}$$

$$F_7=\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijkp}}\left(\left(f_{\bar{x}x}(x_i,y,z,t)-f_{\bar{x}xijkp}\right)\left(f-f(x_i,y,z,t)\right)+\right. \tag{75}$$

$$+\left(f_{\bar{y}y}(x,y_j,z,t)-f_{\bar{y}yijkp}\right)\left(f-f(x,y_j,z,t)\right)+$$

$$+\left(f_{\bar{z}z}(x,y,z_k,t)-f_{\bar{z}zijkp}\right)\left(f-f(x,y,z_k,t)\right)\Big)\,dQ_{ijkp}$$

$$F_8=-\sum_{p=1}^{P}\int_{t_{p-1}}^{t_p}\left(\sum_{j,k=1}^{\overline{Y_h},\overline{Z_h}}\int_{Q_{jk}}\left(f_{\bar{x}}(x_{X_h},y,z,t)-f_{\bar{x}X_hjkp}\right)\left(f(x_{\overline{X_h}},y,z,t)-\right.\right.$$
$$\left.-f_{\overline{X_h}jkp}\right)dQ_{jk}+\sum_{i,k=1}^{\overline{X_h},\overline{Z_h}}\int_{Q_{ik}}\left(f_{\bar{y}}(x,y_{Y_h},z,t)-f_{yiY_hkp}\right)\left(f(x,y_{\overline{Y_h}},z,t)-\right. \tag{76}$$
$$\left.-f_{i\overline{Y_h}kp}\right)dQ_{ik}+\sum_{i,j=1}^{\overline{X_h},\overline{Y_h}}\int_{Q_{ij}}\left(f_{\bar{z}}(x,y,z_{Z_h},t)-f_{yijZ_hp}\right)\left(f(x,y,z_{\overline{Z_h}},t)-\right.$$
$$\left.\left.-f_{ij\overline{Z_h}p}\right)dQ_{ij}\right)dt$$

$$F_9=-\sum_{p=1}^{P}\int_{t_{p-1}}^{t_p}\left(\sum_{j,k=1}^{\overline{Y_h},\overline{Z_h}}\int_{Q_{jk}}\left(f_{\bar{x}}(x_1,y,z,t)-f_{\bar{x}1jkp}\right)\left(f(0,y,z,t)-\right.\right.$$
$$\left.-f_{0jkp}\right)dQ_{jk}+\sum_{i,k=1}^{\overline{X_h},\overline{Z_h}}\int_{Q_{ik}}\left(f_{\bar{y}}(x,y_1,z,t)-f_{yi1kp}\right)\left(f(x,0,z,t)-\right. \tag{77}$$
$$\left.-f_{i0kp}\right)dQ_{ik}+\sum_{i,j=1}^{\overline{X_h},\overline{Y_h}}\int_{Q_{ij}}\left(f_{\bar{z}}(x,y,z_1,t)-f_{yij1p}\right)\left(f(x,y,0,t)-\right.$$
$$\left.\left.-f_{ij0p}\right)dQ_{ij}\right)dt$$

$$F_{10}=\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijkp}}\left(u-u_{ijkp}\right)\left(f-f_{ijkp}\right)dQ_{ijkp} \tag{78}$$

Before estimating $|F_i|$, it is necessary to introduce some more auxiliary inequalities.

If we take the function $f_{\bar{x}}(x_{p+1},y,z,t)$ and its analogue $f_{\bar{x}ijkp}$, is discrete, expanding them with respect to the variable x, and summing from$m$ to i, we get the following 2 equalities:

$$f(x_i,y,z,t)=\sum_{n=m+1}^{i}h_X f_{\bar{x}}(x_n,y,z,t)+f(x_m,y,z,t),$$
$$\forall(y,z,t)\in[0,l_y]\times[0,l_z]\times[0,T] \tag{79}$$
$$f_{ijkp}=\sum_{n=m+1}^{i}h_X f_{\bar{x}njkp}+f_{mjkp}$$
$$\forall j=1,\dots,\overline{Y_h},\forall k=1,\dots,\overline{Z_h},\forall p=1,\dots,P$$

Where $1\le a\le i\le\overline{X_h}$; for $i=a$ y definition, we consider the sum in any of the equalities to be 0.

If we subtract equalities (79) from each other, square both sides, and use the elementary inequality from (22), then we have the inequality:

$$\left(f(x_i,y,z,t)-f_{ijkp}\right)^2\le 2\left(\left(f(x_m,y,z,t)-f_{mjkp}\right)^2+\right.$$
$$\left.+\sum_{n=m+1}^{i}h_X\left(f_{\bar{x}}(x_a,y,z,t)-h_X f_{\bar{x}ajkp}\right)^2\right) \tag{80}$$

$\forall(y,z,t) \in [0,l_y] \times [0,l_z] \times [0,T]$,

$\forall j = 1,\dots,Y_h, \forall k = 1,\dots,Z_h, \forall p = 1,\dots,P$

We also note that:

$$\sum_{p,a,b,c=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{abcp}} \left(f(x_a,y,z,t) - f_{abcp}\right)^2 dQ_{abcp} = \sum_{p,a,b,c=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{abcp}} \Big((f(x_i,y,z,t) - f) + \big(f - f(x,y,z,t_p)\big) +$$

$$+\big(f(x,y,z,t_p) - f_{abcp}\big)\Big)^2 dQ_{abcp} \leq$$

Using the elementary inequality from (22) for the square of the trinomial, as well as the property that the square of the integral does not exceed the integral of the square, we pass to the inequality:

$$\leq 3 \sum_{p,a,b,c=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{abcp}} \left(\int_{x_a}^{x_{a+1}} h_X \left(\frac{\partial f}{\partial x}\right)^2 d\xi + \int_{t_{\overline{p}}}^{t_p} \tau \left(\frac{\partial f}{\partial t}\right)^2 d\eta + \right.$$

$$\left. + f(x,y,z,t_p) - f_{abcp}\right) dQ_{abcp} \leq 3 \left(h_X^2 \left\| \frac{\partial f}{\partial x} \right\|_{L_2(Q)}^2 + \tau^2 \left\| \frac{\partial f}{\partial t} \right\|_{L_2(Q)}^2 + \sum_{p,a,b,c=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{abc}} \tau\big(f(x,y,z,t_p) - f_{abcp}\big) dQ_{abc} \right)$$

If we write down similar estimates for the spatial variables y and z, then add them up and apply estimate (39), we obtain:

$$\sum_{p,a,b,c=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{abcp}} \Big(\left(f(x_a,y,z,t) - f_{abcp}\right)^2 + \left(f(x,y_b,z,t) - f_{abcp}\right)^2 +$$

$$+\left(f(x,y,z_c,t) - f_{abcp}\right)^2 \Big) dQ_{abcp} \leq C(h_X^2 + h_Y^2 + h_Z^2 + 3\tau^2) \cdot \qquad (81)$$

$$\cdot \|u\|_{L_2(Q)}^2 + \sum_{p,a,b,c=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{abc}} 3\tau\big(f(x,y,z,t_p) - f_{abcp}\big) Q_{abc}$$

Based on formulas (80)-(81) it follows that

$$\sum_{p,j,k=1}^{P,\overline{Y_h},\overline{Z_h}} \int_{Q_{jkp}} \left(f(x_i,y,z,t) - f_{ijkp}\right)^2 dQ_{jkp} +$$

$$+ \sum_{p,i,k=1}^{P,\overline{X_h},\overline{Z_h}} \int_{Q_{ikp}} \left(f(x,y_j,z,t) - f_{ijkp}\right)^2 dQ_{ikp} +$$

$$+ \sum_{p,i,j=1}^{P,\overline{X_h},\overline{Y_h}} \int_{Q_{ijp}} \left(f(x,y,z_k,t) - f_{ijkp}\right)^2 dQ_{ijp} \leq \qquad (82)$$

$$\leq C \left(\sum_{p,a,b,c=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{abc}} \left(\int_{t_{p-1}}^{t_p} \Big(\left(f_{\bar{x}}(x_a,y,z,t) - f_{\bar{x}abcp}\right)^2 + \right.\right.$$

$$+\left(f_{\bar{z}}(x,y,z_c,t) - f_{\bar{z}abcp}\right)^2 + \left(f_{\bar{y}}(x,y_b,z,t) - f_{\bar{y}abcp}\right)^2 \Big) dt +$$

$$\left.\left. + 3\tau\big(f(x,y,z,t_p) - f_{abcp}\big)^2 \right) dQ_{abc} + C(h_X + h_Y + h_Z + 3\tau) \|u\|_{L_2(Q)}^2\right.$$

$i = 1,\dots,X_h, j = 1,\dots,Y_h, k = 1,\dots,Z_h$

Further note that

$$\frac{\partial f}{dx} - f_{\bar{x}}(x_i, y, z, t) = \frac{1}{h_X} \int_{x_{i-1}}^{x_i} \left(\frac{\partial f}{\partial x} - \frac{\partial f(\xi, y, z, t)}{\partial x} \right) d\xi =$$

$$= \frac{1}{h_X} \int_{x_{i-1}}^{x_i} \left(\int_{\xi}^{x} \frac{\partial^2 f(\eta, y, z, t)}{\partial x^2} d\eta \right) d\xi \tag{83}$$

$$\forall x \in [0, l_x], (y, z, t) \in [0, l_y] \times [0, l_z] \times [0, T], i = 1, \dots, X_h$$

Hence, for all $s, s_{i-1} \le s \le s_{i+1}, i = 1, \dots, \overline{X_h}$, we have:

$$\sum_{p,j,k=1}^{P,\overline{Y_h},\overline{Z_h}} \int_{Q_{jkp}} \left(\frac{\partial f}{dx} - f_{\bar{x}}(x_i, y, z, t) \right)^2 dQ_{jkp} \le$$

$$\le \sum_{p,j,k=1}^{P,\overline{Y_h},\overline{Z_h}} \int_{Q_{jkp}} \left(\frac{1}{h_X} \int_{x_{i-1}}^{x_i} \left(\int_{x_{i-1}}^{x_{i+1}} \left| \frac{\partial^2 f(\eta, y, z, t)}{\partial x^2} \right| d\eta \right) d\xi \right)^2 dQ_{jkp} \le \tag{84}$$

$$\le 2h_X \sum_{p,j,k=1}^{P,\overline{Y_h},\overline{Z_h}} \int_{Q_{jkp}} \int_{x_{i-1}}^{x_{i+1}} \left| \frac{\partial^2 f(\eta, y, z, t)}{\partial x^2} \right|^2 d\eta \, dQ_{jkp}$$

If you perform similar operations for other spatial variables, then add up the estimates and change the integration region from the interval to the entire region$[0, l_x] \times [0, l_y] \times [0, l_z] \times [0, T]$, and use estimate (39), and for h_{max} take $\max\{h_X, h_Y, h_Z\}$, then we pass to the following inequality:

$$\sum_{p,j,k=1}^{P,\overline{Y_h},\overline{Z_h}} \int_{Q_{jkp}} \left(\frac{\partial f}{dx} - f_{\bar{x}}(x_i, y, z, t) \right)^2 dQ_{jkp} +$$

$$+ \sum_{p,i,k=1}^{P,\overline{X_h},\overline{Z_h}} \int_{Q_{ikp}} \left(\frac{\partial f}{dy} - f_{\bar{y}}(x, y_j, z, t) \right)^2 dQ_{ikp} + \tag{85}$$

$$+ \sum_{p,i,j=1}^{P,\overline{X_h},\overline{Y_h}} \int_{Q_{ijp}} \left(\frac{\partial f}{dz} - f_z(x, y, z_k, t) \right)^2 dQ_{ijp} \le$$

$$\le 2h_{\max} \left(\left\| \frac{\partial^2 f}{\partial x^2} \right\|_{L_2(Q)}^2 + \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{L_2(Q)}^2 + \left\| \frac{\partial^2 f}{\partial z^2} \right\|_{L_2(Q)}^2 \right) \le h_{max} C \|u\|_{L_2(Q)}^2$$

In addition, taking into account (84), (39) by performing similar operations, we can obtain:

$$\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijkp}} \left(\left(\frac{\partial f}{dx} - f_{\bar{x}}(x_i, y, z, t) \right)^2 + \left(\frac{\partial f}{dy} - f_{\bar{y}}(x, y_j, z, t) \right)^2 + \right.$$

$$\left. + \left(\frac{\partial f}{dz} - f_z(x, y, z_k, t) \right)^2 \right) dQ_{ijkp} \le \tag{86}$$

$$\le 2h_{max}^2 \left(\left\| \frac{\partial^2 f}{\partial x^2} \right\|_{L_2(Q)}^2 + \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{L_2(Q)}^2 + \left\| \frac{\partial^2 f}{\partial z^2} \right\|_{L_2(Q)}^2 \right) \le h_{max}^2 C \|u\|_{L_2(Q)}^2$$

If we write the function $f_{\bar{x}x}(x_i, y, z, t)$ in x_i and go from the difference via Newton-Leibniz back to the integral, and also use the estimate (39), then we can obtain the following estimate:

$$\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijkp}} \left(f_{\bar{x}x}(x_i, y, z, t) \right)^2 dQ_{ijkp} =$$

$$= \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijkp}} \frac{1}{h_X^4}\left(\int_{x_i}^{x_{i+1}}\left(\int_{\xi-h_X}^{\xi}\frac{\partial^2 f(\eta,y,z,t)}{\partial x^2}d\eta\right)d\xi\right)^2 dQ_{ijkp} \leq$$

$$\leq 2\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{jkp}} \int_{x_{i-1}}^{x_{i+1}} \left|\frac{\partial^2 f(\eta,y,z,t)}{\partial x^2}\right|^2 d\eta\, dQ_{jkp} \leq 2\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{L_2(Q)}^2$$

If we perform the mathematical transformations for the functions $f_{\bar{y}y}(x_i,y,z,t)$ and $f_{\bar{y}y}(x_i,y,z,t)$, in a similar way, and then sum them all up, we can obtain the inequality:

$$\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijkp}} \left(\left(f_{\bar{x}x}(x_i,y,z,t)\right)^2 + \left(f_{\bar{x}x}(x_i,y,z,t)\right)^2 + + \left(f_{\bar{x}x}(x_i,y,z,t)\right)^2\right) dQ_{ijkp} \leq C\|u\|_{L_2(Q)}^2 \tag{87}$$

Now we can proceed to estimates of the quantities $F_i, i = 1, \dots, 10$ from (68). Let's start by evaluating F_1. To do this, we pass from the difference, through Newton - Leibniz, to integration, use the elementary inequality from (22) for the product, and also take into account estimates (39) and (58):

$$|F_1| \leq \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijkp}} \left(\int_{t_{p-1}}^{t_p}\frac{\partial f(x,y,z,\tau)}{\partial t}d\tau\right) f_{\bar{t}ijkp}\, dQ_{ijkp} \leq \leq \frac{1}{2}\tau\left(\left\|\frac{\partial f}{\partial t}\right\|_{L_2(Q)}^2 + \left\|f_{\bar{t}ijkp}\right\|_{L_{2h\tau}}^2\right) \leq \tau C(\|u\|_{L_2(Q)}^2 + \|u\|_{L_{2h\tau}}^2) \tag{88}$$

To estimate F_2, using the Cauchy-Bunyakovsky formula (21), we represent the sums of the products in the form of the product of the sums, after which we apply the elementary inequality from (22) for the product, and estimate the resulting terms using formulas (82) and (85). We get the following estimate:

$$|F_2| \leq \frac{\varepsilon}{2}C\left(\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} \left(\int_{t_{p-1}}^{t_p}\left(\left(f_{\bar{x}}(x_i,y,z,t) - f_{\bar{x}ijkp}\right)^2 + +\left(f_{\bar{y}}(x,y_j,z,t) - f_{\bar{y}ijkp}\right)^2 + \left(f_{\bar{z}}(x,y,z_k,t) - f_{\bar{z}ijkp}\right)^2\right)dt + +3\tau\left(f(x,y,z,t_p) - f_{ijkp}\right)^2\right) dQ_{ijk} + +C(h_X + h_Y + h_Z + 3\tau)\|u\|_{L_2(Q)}^2\right) + \frac{h_{\max}}{2\varepsilon}C\|u\|_{L_2(Q)}^2 \tag{89}$$

Similarly to the estimate F_2, we obtain the estimate F_3:

$$|F_3| \leq \frac{\varepsilon}{2}C\left(\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} \left(\int_{t_{p-1}}^{t_p}\left(\left(f_{\bar{x}}(x_i,y,z,t) - f_{\bar{x}ijkp}\right)^2 + +\left(f_{\bar{y}}(x,y_j,z,t) - f_{\bar{y}ijkp}\right)^2 + \left(f_{\bar{z}}(x,y,z_k,t) - f_{\bar{z}ijkp}\right)^2\right)dt + +3\tau\left(f(x,y,z,t_p) - f_{ijkp}\right)^2\right) dQ_{ijk} + + C(h_X + h_Y + h_Z + 3\tau)\left||u|\right|_{L_2(Q)}^2\right) + \frac{h_{\max}}{2\varepsilon}C\|u\|_{L_2(Q)}^2 \tag{90}$$

To estimate F_4, we break each term by the Cauchy-Bunyakovsky formula (21). In each case, we reduce the second factor to the control norm in the space $L_{2h\tau}$ taking into account formula (58). In the first factor, add and subtrac $f_x(x_{i+1},y,z,t), f_y(x,y_{j+1},z,t), f_z(x,y,z_{k+1},t)$ in accordance with the spatial variable, we apply the elementary inequality from (22) for the square of the trinomial, we use estimates (39), (86) and (87). We obtain the following inequality:

$$|F_4| \le Ch_{\max}\|u\|_{L_2(Q)}\|u_{h\tau}\|_{L_{2h\tau}} \le h_{\max}C\left(\|u\|^2_{L_2(Q)} + \|u_{h\tau}\|^2_{L_{2h\tau}}\right) \tag{91}$$

To estimate F_5, we break each term by the Cauchy-Bunyakovsky formula (21), and then evaluate them individually using formulas (34) and (86). We have:

$$|F_5| \le h_{\max}C\|u\|^2_{L_2(Q)} \tag{92}$$

To estimate F_6, we divide each term by the Cauchy-Bunyakovsky formula (21), integrate the first factor of each term with respect to the corresponding spatial variable, and use estimates (34), (39) and (87). We have:

$$|F_6| \le h_{\max}C\|u\|^2_{L_2(Q)} \tag{93}$$

To estimate F_7, we use the Cauchy-Bunyakovsky formula (21). The left factor is estimated using (39),(58) and (87), in the right we pass from the difference to integration, apply the Cauchy-Bunyakovsky formula (21) and evaluate it using the formula (34). We have:

$$|F_7| \le h_{\max}C\left(\|u\|^2_{L_2(Q)} + \|u_{h\tau}\|^2_{L_{2h\tau}}\right) \tag{99}$$

To estimate F_8 and F_9, we note that $f_{\bar{x}}(x_{X_h}, y, z, t) - f_{\bar{x}X_h jkp} = f_{\bar{x}}(x_{X_h}, y, z, t) - \partial f_{\bar{x}}(x_{X_h}, y, z, t)/\partial x$ and $f_{\bar{x}}(x_1, y, z, t) - f_{\bar{x}1jkp} = f_{\bar{x}}(x_1, y, z, t) - \partial f_{\bar{x}}(x_1, y, z, t)/\partial x$ due to the boundary conditions (5) and (19) both for the variable x, and for other spatial variables y and z. Given these conditions, we estimate F_8 and F_9 similarly to the estimates F_2 and F_3:

$$\begin{aligned}|F_8| \le \frac{\varepsilon}{2}C\Bigg(\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}}\Bigg(\int_{t_{p-1}}^{t_p}\Big(\left(f_{\bar{x}}(x_i, y, z, t) - f_{\bar{x}ijkp}\right)^2 + \\ +\left(f_{\bar{y}}(x, y_j, z, t) - f_{\bar{y}ijkp}\right)^2 + \left(f_{\bar{z}}(x, y, z_k, t) - f_{\bar{z}ijkp}\right)^2\Big)dt + \\ +3\tau\left(f(x, y, z, t_p) - f_{ijkp}\right)^2\Bigg)dQ_{ijk} + \\ + C(h_X + h_Y + h_Z + 3\tau)\|u\|^2_{L_2(Q)}\Bigg) + \frac{h_{\max}}{2\varepsilon}C\|u\|^2_{L_2(Q)}\end{aligned} \tag{100}$$

$$\begin{aligned}|F_9| \le \frac{\varepsilon}{2}C\Bigg(\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}}\Bigg(\int_{t_{p-1}}^{t_p}\Big(\left(f_{\bar{x}}(x_i, y, z, t) - f_{\bar{x}ijkp}\right)^2 + \\ +\left(f_{\bar{y}}(x, y_j, z, t) - f_{\bar{y}ijkp}\right)^2 + \left(f_{\bar{z}}(x, y, z_k, t) - f_{\bar{z}ijkp}\right)^2\Big)dt + \\ +3\tau\left(f(x, y, z, t_p) - f_{ijkp}\right)^2\Bigg)dQ_{ijk} + \\ + C(h_X + h_Y + h_Z + 3\tau)\|u\|^2_{L_2(Q)}\Bigg) + \frac{h_{\max}}{2\varepsilon}C\|u\|^2_{L_2(Q)}\end{aligned} \tag{101}$$

To evaluate F_{10}, we use Cauchy-Bunyakovsky (21) and (39):

$$\begin{aligned}|F_{10}| = \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijkp}} (u - u_{ijkp})(f - f_{ijkp})dQ_{ijkp} \le \\ \le \frac{1}{2}\|u - b_{h\tau}u_{h\tau}\|^2_{L_2(Q_h)} + \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijkp}}\left(\left(\int_{t_{p-1}}^{t_p}\frac{\partial f(x, y, z, \tau)}{\partial t}d\tau\right)^2 + \right. \\ \left. +\left(f(x, y, z, t_p) - f_{ijkp}\right)^2 dQ_{ijkp}\right) \le \frac{1}{2}\|u - b_{h\tau}u_{h\tau}\|^2_{L_2(Q_h)} + \\ +\tau^2C\|u\|^2_{L_2(Q)} + \tau\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} \left(f(x, y, z, t_p) - f_{ijkp}\right)^2 dQ_{ijk}\end{aligned} \tag{102}$$

We substitute all the obtained estimates (88)-(102) into the inequality (68):

$$\frac{1}{2}\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left|f(x,y,z,t_P)-f_{ijkP}\right|^2 dQ_{ijk}$$

$$+\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijkp}}\left(\left(f(x_i,y,z,t)-f_{\bar{x}ijkp}\right)^2+\left(f(x,y_j,z,t)-f_{\bar{y}ijkp}\right)^2+\right.$$

$$\left.+\left(f(x,y,z_k,t)-f_{\bar{z}ijkp}\right)^2\right)dQ_{ijkp}\le\|u-b_{h\tau}u_{h\tau}\|^2_{L_2(Q_h)}+$$

$$+C(h_X+h_Y+h_Z+3\tau)\left(1+\varepsilon+\frac{1}{\varepsilon}\right)\left(\|u\|^2_{L_2(Q)}+\|u\|^2_{L_{2h\tau}}\right)+$$

$$+\tau(1+3\varepsilon C)\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left(f(x,y,z,t_p)-f_{ijkp}\right)^2 dQ_{ijk}+$$

$$+\varepsilon C\left(\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left(\int_{t_{p-1}}^{t_p}\left(\left(f_{\bar{x}}(x_i,y,z,t)-f_{\bar{x}ijkp}\right)^2+\left(f_{\bar{y}}(x,y_j,z,t)-f_{\bar{y}ijkp}\right)^2+\right.\right.\right.$$

$$\left.\left.\left.+\left(f_{\bar{z}}(x,y,z_k,t)-f_{\bar{z}ijkp}\right)^2\right)dt\right)dQ_{ijk}\right)$$

Or, if we group the elements, we get:

$$\left(\frac{1}{2}-\tau(1+3\varepsilon C)\right)\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left|f(x,y,z,t_P)-f_{ijkP}\right|^2 dQ_{ijk}+(1-\varepsilon C)\cdot$$

$$\cdot\sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijkp}}\left(\left(f(x_i,y,z,t)-f_{\bar{x}ijkp}\right)^2+\left(f(x,y_j,z,t)-f_{\bar{y}ijkp}\right)^2+\right.$$

$$\left.+\left(f(x,y,z_k,t)-f_{\bar{z}ijkp}\right)^2\right)dQ_{ijkp}\le\|u-b_{h\tau}u_{h\tau}\|^2_{L_2(Q_h)}+ \tag{103}$$

$$+\tau(1+3\varepsilon C)\sum_{p,i,j,k=1}^{\bar{P},\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left(f(x,y,z,t_p)-f_{ijkp}\right)^2 dQ_{ijk}+$$

$$+C(h_X+h_Y+h_Z+3\tau)\left(1+\varepsilon+\frac{1}{\varepsilon}\right)\left(\|u\|^2_{L_2(Q)}+\|u\|^2_{L_{2h\tau}}\right)$$

We introduce some conditions. We fix $\varepsilon>0$ so small that the inequality $1-\varepsilon C>0$ holds. In addition, we assume that τ is so small that $1/2-\tau(1+3\varepsilon C)\ge 1/4$. Then from inequality (103) we come to:

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left|f(x,y,z,t_P)-f_{ijkP}\right|^2 dQ_{ijk}\le\|u-b_{h\tau}u_{h\tau}\|^2_{L_2(Q_h)}+$$

$$+4\tau(1+3\varepsilon C)\sum_{p,i,j,k=1}^{\bar{P},\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left(f(x,y,z,t_p)-f_{ijkp}\right)^2 dQ_{ijk}+ \tag{104}$$

$$+C(h_X+h_Y+h_Z+3\tau)\left(1+\varepsilon+\frac{1}{\varepsilon}\right)\left(\|u\|^2_{L_2(Q)}+\|u\|^2_{L_{2h\tau}}\right)$$

For inequality (104), we use Lemma 1, introduce the constants $C_1=4(1+3\varepsilon C)$ and $C_2=C(1+\varepsilon+1/\varepsilon)$. We have:

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} \left|f(x,y,z,t_P) - f_{ijkP}\right|^2 dQ_{ijk} \leq (1+\tau C_1)\cdot$$
$$\cdot\left(\|u - b_{h\tau}u_{h\tau}\|^2_{L_2(Q_h)} + C_2(h_X + h_Y + h_Z + 3\tau)\left(\|u\|^2_{L_2(Q)} + \|u\|^2_{L_{2h\tau}}\right)\right) \tag{105}$$

Note that when expanding the function $e^{\tau C_1}$ in a Taylor series, the first 2 terms of the expansion correspond to $1+\tau C_1$, whence we can obtain the inequality $1+\tau C_1 \leq e^{\tau C_1}$, and it follows $(1+\tau C_1)^P \leq e^{TC_1}$. Based on this:

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} \left|f(x,y,z,t_P) - f_{ijkP}\right|^2 dQ_{ijk} \leq$$
$$\leq C\left(\|u - b_{h\tau}u_{h\tau}\|^2_{L_2(Q_h)} + (h_X + h_Y + h_Z + 3\tau)\left(\|u\|^2_{L_2(Q)} + \|u\|^2_{L_{2h\tau}}\right)\right) \tag{106}$$

Using estimate (106), we prove that problem (17)-(19) approximates problem (3)-(6) with respect to function.

Theorem 2. Let the step function correspond to problem (17)-(19):

$$b_{ijk} = \frac{1}{h}\int_{Q_{ijk}} b(\xi,\eta,\phi)d\xi d\eta d\phi,$$
$$i = \overline{1..X_h - 1}, j = \overline{1..Y_h - 1}, k = \overline{1..Z_h - 1} \tag{107}$$

Then $\lim_{(h_X,h_Y,h_Z,\tau)\to 0} J_{h\tau*} = J_*$.

Let us prove this theorem. Let us evaluate the difference $J(u) - J_{h\tau}(u_{h\tau})$, assuming that $u \in U$, $u_{h\tau} \in U_{h\tau}$. Taking into account estimates (34),(39),(49),(58) and (106), the definition of the sets (3) and (17), the inequality;

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} hb^2_{ijk} = \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} h\left(\frac{1}{h}\int_{Q_{ijk}} b(\xi,\eta,\phi)d\xi d\eta d\phi\right)^2 \leq \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} b^2(\xi,\eta,\phi)d\xi d\eta d\phi \leq \|b\|^2_{L_2(\overline{Q_3})} \tag{108}$$

Then if we subtract the discrete criterion (4) from the criterion of the differential problem (18) and estimate the difference:

$$|J(u) - J_{h\tau}(u_{h\tau})| = \int_0^h |f(x,y,z,T;u) - b(x,y,z)|^2 dQ_3 +$$
$$+\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} \left(\left(f(x,y,z,t_P;u) - f_{ijkP}\right) + \left(b(x,y,z) - b_{ijk}\right)\right)\cdot$$
$$\cdot\left(f(x,y,z,t_P;u) + b_{ijk} - f_{ijkP} - b(x,y,z)\right)dQ_3 \leq$$
$$\leq 2\left(\int_0^h \left(f^2(x,y,z,T;u) - b^2(x,y,z)\right)dQ_3\right) +$$
$$\left[\left(\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} \left(f(x,y,z,t_P;u) - f_{ijkP}\right)^2 dQ_3\right)^{\frac{1}{2}} +\right.$$
$$\left.+\left(\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} \left(b(x,y,z) - b_{ijk}\right)^2 dQ_3\right)^{\frac{1}{2}}\right] \times \left[\left(\int_{Q_3} f^2(x,y,z,t_P;u)dQ_3\right)^{\frac{1}{2}} +\right.$$

$$+\left(\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} hf_{ijkP}^2\right)^{\frac{1}{2}} + \|b\|_{L_2(\overline{Q_3})}^2 + \left(\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} hb_{ijk}^2\right)^{\frac{1}{2}}\Bigg] \leq$$

$$\leq 2\int_0^h b^2(x,y,z)dQ_3 + C\left(\|u - b_{h\tau}u_{h\tau}\|_{L_2(Q_h)} + \sqrt{(h_X + h_Y + h_Z + 3\tau)} +\right.$$

$$\left.+hCR^2 + \left(\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} \left(b(x,y,z) - b_{ijk}\right)^2 dQ_3\right)^{\frac{1}{2}}\right)$$

We can obtain the following inequality:

$$|J(u) - J_{h\tau}(u_{h\tau})| \leq hCR^2 + 2\int_0^h b^2(x,y,z)dQ_3 +$$
$$+C\left(\|u - b_{h\tau}u_{h\tau}\|_{L_2(Q_h)} + \sqrt{(h_X + h_Y + h_Z + 3\tau)} +\right. \tag{109}$$
$$\left.+\left(\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} \left(b(x,y,z) - b_{ijk}\right)^2 dQ_3\right)^{\frac{1}{2}}\right)$$

We estimate the differential function and its step analogue to show that as $(h_X, h_Y, h_Z) \to 0$ due to the average continuity of the function $b(x,y,z) \in L_{2(Q_3)}$, its square is the difference with the step function b_{ijk} tends to 0. For this, we integrate the square of the difference of functions over the variables and use the definition (107), we obtain:

$$\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} \left(\frac{1}{h}\int_{Q_{ijk}} \left(b(x,y,z) - b(\xi,\eta,\phi)\right)d\xi d\eta d\phi\right)^2 dQ_{ijk} \leq$$
$$\leq \sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} \left(\frac{1}{h}\int_{-h}^{h} (\Delta b)^2 dadcde\right) dQ_{ijk} = \tag{110}$$
$$= \frac{1}{h}\int_{-h}^{h} \left(\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijk}} (\Delta b)^2 dQ_{ijk}\right) dadcde \leq \max_{|(a,b,c)|\leq h} \int_{Q_3} |\Delta b|^2 dQ_3 \to 0,$$
$$\Delta b = b(x+a, y+c, z+e) - b(x,y,z)$$

Since problem (5), (6) and (19) have a solution, i.e., $U_* \neq \emptyset, U_{h\tau*} \neq \emptyset$, we fix some u_* and $u_{h\tau*}$, so that $u_* \in U_*, u_{h\tau*} \in U_{h\tau*}$. Since $\|b_{h\tau}u_{h\tau*}\|_{L_2(Q)} = \|u_{h\tau*}\|_{L_{2h\tau}} \leq R$, taking $b_{h\tau}u_{h\tau*} = 0$ outside Q_h, we can assume that $b_{h\tau}u_{h\tau*} \in U$. For the control function $u_* \in U_*$ we construct its discrete analog $Q_{h\tau}u_* = \{u_{*ijkp}, i = \overline{1..\overline{X_h}}, j = \overline{1..\overline{Y}_h}, k = \overline{1..\overline{Z_h}}, p = \overline{1..P}\}$ by the rule:

$$u_{*ijkp} = \frac{1}{h\tau}\int_{Q_{ijkp}} u_*(x,y,z,t)dQ_{ijkp} \tag{111}$$

We show that $Q_{h\tau}u_* \in U_{h\tau}$. For this we show:

$$\|Q_{h\tau}u_*\|_{L_{2h\tau}}^2 \leq \sum_{p,i,j,k=1}^{P,\overline{X_h},\overline{Y_h},\overline{Z_h}} \int_{Q_{ijkp}} u_*^2(x,y,z,t)dQ_{ijkp} \leq \|u_*\|_{L_2(Q)} \leq R \tag{112}$$

Having performed mathematical transformations similar to formula (110), we obtain:

$$\lim_{(h_X,h_Y,h_Z,\tau)\to 0}\|u_*(x,y,z,t)-b_{h\tau}Q_{h\tau}u_*\|_{L_{2(Q_h)}}=0 \tag{113}$$

We find the upper and lower limits of the difference in the criteria of the differential and discrete problems in order to determine an estimate of the rate of convergence. Let's start with the upper limit:

$$J_*-J_{h\tau*}\le J(b_{h\tau}u_*)-J_{h\tau}(u_{h\tau*})\le hCR^2+C\sqrt{(h_X+h_Y+h_Z+3\tau)}+$$
$$+2\int_0^h b^2(x,y,z)dQ_3+C\left(\left(\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left(b(x,y,z)-b_{ijk}\right)^2dQ_3\right)^{\frac{1}{2}}\right) \tag{114}$$

$$\overline{\lim_{(h_X,h_Y,h_Z,\tau)\to 0}}(J_*-J_{h\tau*})\le 0 \tag{115}$$

We begin by estimating the last term from the right-hand side. Based on the formula (112), we discard it, since it tends to 0 with decreasing step. The second term with a sufficiently small difference in the arguments of a certain integral will give an insignificant value. The first term, in comparison with the third, has a larger order of smallness; therefore, the upper limit is determined by such a quantity as $C\sqrt{(h_X+h_Y+h_Z+3\tau)}$.

We perform the same operations to determine the lower limit, but change the arguments for the criteria of the differential and discrete problems. We get:

$$J_*-J_{h\tau*}\ge J(u_*)-J_{h\tau}(Q_{h\tau}u_*)\ge -C\sqrt{(h_X+h_Y+h_Z+3\tau)}-$$
$$-C\left(\left(\sum_{i,j,k=1}^{\overline{X_h},\overline{Y_h},\overline{Z_h}}\int_{Q_{ijk}}\left(b(x,y,z)-b_{ijk}\right)^2dQ_3\right)^{\frac{1}{2}}\right)-2\int_0^h b^2(x,y,z)dQ_3- \tag{116}$$
$$-\|u_*(x,y,z,t)-b_{h\tau}Q_{h\tau}u_*\|_{L_{2(Q_h)}}-hCR^2$$

$$\underline{\lim_{(h_X,h_Y,h_Z,\tau)\to 0}}(J_*-J_{h\tau*})\le 0 \tag{117}$$

Based on the formula (113), the first term is neglected due to its lesser influence on the right side in comparison with others. For the remaining elements of the right-hand side, the conclusions remain similar to the conclusions for the formula (114).

The limits of (115), (117) imply the statement of Theorem 2. Inequalities (114), (116) estimate the rate of convergence for $J_*-J_{h\tau*}$. If $b(x,y,z), u_*=u_*(x,y,z,t)$ are sufficiently smooth, then it follows from (114), (116) that:

$$|J_*-J_{h\tau*}|=O\left(\sqrt{(h_X+h_Y+h_Z+3\tau)}\right) \tag{118}$$

9. Conclusion

The problem of determining the approximation order of the optimal control problem for the spatial process of heat conduction is considered in the paper. Using the methods of integral inequalities and the method of difference approximation, a difference problem is obtained, an algorithm for finding its solution is described, and an estimate is obtained for the deviation of the value of the difference functional from the continuous functional. The established inequality Equation (118) gives an idea of the time complexity of the process of calculating an approximate solution when the accuracy of calculations is given in advance. The time steps and spatial variables are not independent, additional restrictions must be imposed to ensure stability. The methodology for obtaining an approximation estimate can be used for implicit approximations, hybrid schemes. The methods used in this article can be successfully applied for similar parabolic problems with bounded coefficients, as well as for problems of large dimensions.

10. Conflicts of Interest

The authors declare no conflict of interest.

11. References

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